Technical Report No. AF-94

SPECTRAL CHARACTERISTICS OF RANDOM MODULATED WAVES

by

L. K. Lauderdale

THE JOHNS HOPKINS UNIVERSITY
RADIATION LABORATORY
BALTIMORE 2, MD.

CONTRACT NO. AF 33(616)-6753

MAY 1962

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ACKNOWLEDGMENT

During the course of carrying out the investigation leading to this report and in the preparation of the manuscript, the author was very fortunate in receiving extremely beneficial cooperation and support from staff members of The Johns Hopkins University Electrical Engineering Department and the Radiation Laboratory. The author wishes to express his sincere gratitude to all those who have contributed to this report and, in particular, to Professors W. C. Gore and W. H. Huggins, who served as advisors and referees for this work; to Dr. J. M. Kopper, who carefully reviewed the manuscript; to Mr. J. R. Taylor and Mr. L. D. Dickson for the construction of experimental apparatus and the gathering of experimental data; to Mrs. Marguerite Denburg, who performed the computations; to Mr. C. M. LaPorte for drafting services; and to Miss Carole M. Tennant for careful and accurate typing of a difficult manuscript.

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ABSTRACT

The spectral characteristics of random modulated waves are investigated with particular emphasis on the determination of power density spectra of sinusoids that are frequency modulated by band-pass Gaussian noise. Pertinent parts of the theory of random processes necessary for spectral analysis are reviewed, and a general formulation of the problem of determining spectral characteristics of random modulated waves is given. The principal results of this investigation are given in Chapter IV where the variation of FM by band-pass noise power density spectra as a function of the modulation parameters is treated in detail. Limiting cases of large and small modulation indices are discussed, and a technique is developed for obtaining useful approximations of FM by band-pass noise spectra. These results are verified by digital compu' ions and experimental measurements. Finally, spectra resulting from FM by the combination of band-pass noise and a sinusoid, and simultaneous AM by low-pass noise and FM by band-pass noise are investigated.

I. INTRODUCTION

Frequency modulation techniques have found widespread application in electronic communication and, more recently, radar and radar countermeasures systems. It is not surprising, therefore, that considerable effort has been devoted to the study of frequency modulation fundamentals. The development of analytical techniques for determining frequency spectra of frequency modulated waves has been of particular interest, where the modulating waveform may be either periodic or random.

In the case of frequency modulation by random waveforms emphasis has been placed on determining spectra resulting from use of modulating waves that are obtained by filtering white Gaussian noise with a variety of low-pass filters. It is the principal purpose of this investigation to consider modulations which consist of Gaussian noise having a band-pass characteristic; in particular the case of FM by white Gaussian noise that has been filtered by a single stage L-R-C filter will be treated. In addition, a theory for computing spectra resulting from FM by a waveform consisting of a sinusoid and band-pass Gaussian noise combined, and, also, FM by band-pass noise with simultaneous AM by low-pass noise will be given.

It is well known, and almost intuitively obvious, that there exists a limiting condition wherein the spectral shape of an FM waveform will closely resemble the probability density function of the modulating voltage. This limiting condition is associated with an increasing modulation index (ratio of deviation from center frequency to deviation rate). Also it is known that frequency spectra of amplitude modulated (AM) and FM waves are almost identical for very low modulation indices. Therefore, in the case of FM by band-pass noise, an interesting transition in the spectrum occurs as the modulation index is increased, with the spectrum going from a center frequency spike with two sidebands to a Gaussian curve. It will be possible to determine the modulation index required to produce an FM wave whose spectrum exhibits an essentially Gaussian shape when a modulating noise filter of a given Q is employed.

Formal analytical methods of computing FM spectra, in all but the cases of relatively simple modulating waveforms, invariably lead to complicated mathematical expressions which are exceedingly difficult to evaluate explicitly, and give little insight into the interrelationship of the various modulation parameters. Thus one is forced to seek approximate expressions which are capable of giving reasonably accurate results for the parameter range of interest. This state of affairs is not unexpected when dealing with frequency modulation, but is clearly a result of the fact that frequency modulation is a nonlinear process. In

nonlinear analysis approximate solutions are generally the only useful solutions that can be obtained. Of course, in order to have confidence that approximate solutions will provide useful results, it is necessary to determine the conditions (or, equivalently, the range of parameters) under which the approximations are valid; and also, insofar as possible, approximations should be verified by comparison with experimental results obtained from the actual or simulated nonlinear system and by evaluation of exact calculations where the parameters have been fixed to make computation possible. In this investigation of FM spectra, approximations will be supported by both experimentally determined and digitally computed spectral data.

II. A SUMMARY OF THE THEORY OF RANDOM PROCESSES

The determination of spectral characteristics of random modulated waves is essentially an applied problem in the theory of random processes. Accordingly, a summary of applicable portions of the theory will be given in order to provide a basis for a detailed formulation of the power spectrum problem.

Extensive literature exists on the general subject of random (or stochastic) processes with books by Doob (1)[1], and Davenport and Root (2) being notable examples. The former is a rigorous, mathematical text, while the latter contains an applied, engineering treatment of the subject. Throughout the literature there is considerable variation in the terminology and notation that are employed to denote quantities and relationships that are essential for power spectrum computations. The notation introduced in this chapter will be employed throughout the remainder of this report, and defined quantities will be underlined for emphasis.

A. THE RANDOM PROCESS

A <u>random process</u> can be simply defined as a collection (or ensemble) of random time functions. A member function of the ensemble can be written as $x_\alpha(t)$ where t is a continuous parameter,

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[1] Numbers in parentheses refer to references given on page 84.

denoting time, which ranges from $-\infty$ to $+\infty$, and α is a typical point in a probability measure space Ω. A classic example of a random time function that is of central importance in electronics is the noise voltage produced by a resistor due to thermal agitation of electrons. A collection of the noise voltages produced by all possible physical realizations of a resistor of R ohms, for example, would comprise a random process. Since it is not feasible to specify each time function of a random process exactly, various averages must be defined in order to describe the properties that can be expected to apply to an arbitrary member function of the ensemble. This will lead the way toward computing some important properties of the output of a system which has a random input. The "system" that will be considered in this report is an oscillator which has the capability of being modulated in both amplitude and phase, and we shall be interested in studying spectral properties of the output voltage for inputs which consist of various random modulations.

1. Probability Distribution and Density Functions

The random time functions $x_\alpha(t)$ evaluated at a specific time, t_i, comprise a random variable which is defined over the space Ω. (Note: Usage dictates the use of the word variable instead of function as would be more appropriate.) The n^{th} order cumulative probability distribution function of the set

of random variables $x_a(t_1)$, $x_a(t_2), \cdots, x_a(t_n)$ is denoted P_{nf} and defined as

$$P_{nx}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)$$

$$= \text{Prob}\left[\left\{a \,\middle|\, x_a(t_1) \le x_1, \cdots, x_a(t_n) \le x_n\right\}\right]$$

where the second expression should be read as the probability that $x_a(t)$ evaluated at time t_1 will be less than or equal to the value x_1 and simultaneously $x_a(t_2)$ is less than or equal to x_2, and so on to $x_a(t_n) \le x_n$. An equivalent interpretation is that the set of points a which correspond to those member functions of the ensemble that meet the condition $\left[x_a(t_1) \le x_1, \cdots x_a(t_n) \le x_n\right]$ has a measure given by P_{nx}. The function P_{nf} has the usual properties associated with probability; for example, $0 \le P_{nx} \le 1$ and P_{nx} is monotonic nondecreasing. In spectral analysis the second order probability distribution function, P_{2x}, will be of central interest.

The probability density function, W_{nx}, is defined as the mixed partial derivative of P_{nx} when this derivative exists. Thus

$$W_{nx}(x_1, x_2, \cdots x_n; t_1, t_2, \cdots t_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} P_{nx}(x_1, x_2, \cdots x_n; t_1, t_2, \cdots t_n).$$

The joint cumulative probability distribution function of order (nx, my) for the random processes $x_\alpha(t)$ and $y_\alpha(t)$ is given by

$$P_{nx,my}(x_1, \cdots x_n;\ y_1, \cdots, y_m;\ t_1, \cdots, t_n;\ \tau_1, \cdots \tau_m)$$

$$= \text{Prob}\left[\left\{\alpha \,\middle|\, x_\alpha(t_1) \le x_1, \cdots x_\alpha(t_n) \le x_n;\ y_\alpha(\tau_1) \le y_1, \cdots y_\alpha(\tau_m) \le y_m\right\}\right].$$

Two random processes are called independent to order (nx, my) if

$$P_{nx,my}(x_1, \cdots y_n;\ y_1, \cdots, y_m;\ t_1, \cdots, t_n;\ \tau_1, \cdots \tau_m)$$

$$= P_{nx}(x_1, \cdots, x_n;\ t_1, \cdots t_n)\ P_{my}(y_1, \cdots y_m;\ \tau_1, \cdots \tau_m)$$

or, in words, the joint distribution of x and y is equal to the product of the individual distributions.

In view of the fact that the averages that are important in spectral theory can generally be obtained using first and second order distributions, henceforth higher ordered distributions will not be employed in the development to follow. This does not represent any real loss in generality, however, since the extension to higher order statistics is straightforward.

One last definition pertaining to distribution functions and also density functions is the notion of stationarity. A distribution is said to be stationary to second order if the distribution function, P_{2x}, corresponding to $x_\alpha(t)$, is equal to the distribution

corresponding to $x_a(t+\tau)$ where τ is a fixed parameter. That is,

$$P_{2x}(x_1, x_2, t_1, t_2) = P_{2x}(x_1, x_2; t_1 + \tau, t_2 + \tau) \quad .$$

This reduces the number of variables by one since τ can be taken equal to $-t_1$ and thus P_{2x} is a function of x_1, x_2, and $t_2 - t_1$. A similar definition applies for n^{th} order distributions. Also first order stationary distributions are independent of time.

B. STATISTICAL AVERAGES

Since the random process is characterized as a function of two variables, it is apparent that two distinct types of averages are possible. These are: (1) averages over the ensemble variable, a, at a fixed time, t_1; and (2) averages over time for a given member function of the ensemble.

1. Ensemble Averages

The ensemble average, or expected value, of the random variable x is defined as

$$E(x) = \int_{\Omega} x_a \, d\, P_{1x}$$

where E is the expected value operator. Various moments of the random variable x are given by $E(x^n)$. Of principle importance

are the <u>first moment</u> $E(x)$, also called the mean value of and the <u>second moment</u> $E(x^2)$. The <u>variance</u>, denoted as σ^2, is given by

$$\sigma^2 = E\left[(x-E(x))^2\right] \quad .$$

On expanding and carrying out the indicated ensemble averaging this expression reduces to

$$\sigma^2 = E(x^2) - \left[E(x)\right]^2 \quad .$$

Frequently, in order to obtain a less cumbersome notation, a bar will be used to denote an ensemble average, that is,

$$E(x^n) \equiv \overline{x^n} \quad .$$

If the average value of a reasonably well-behaved function[2], f, of the random variable x is desired, the average of f can be formulated in terms of the distribution function of x instead of the distribution function of f, and

$$E\left[f(x_a)\right] = \int_{-\infty}^{\infty} f(x_1)\, d\, P_{1x} \quad .$$

For a justification of this procedure see Davenport and Root (2).

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[2]For example, f is single valued and of bounded variation.

Finally it should be noted that the expected value integral reduces from a Stieltjes integral to an ordinary Riemann integral if P_{1x} has a continuous derivative. That is,

$$E\left[f(x_1)\right] = \int_{-\infty}^{\infty} f(x_1)\, d\, P_{1x} = \int_{-\infty}^{\infty} f(x_1)\, W_{1x}(x_1, t_1)\, dx_1 \quad .$$

Jump discontinuities in the monotonically nondecreasing distribution function P_{1x} can usually be accounted for by using delta functions with the density function, W_{1x}. If g is a function of the two variables x_1 and x_2 then the average of g becomes

$$E\left[g(x_1, x_2)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2)\, W_{2x}(x_1, x_2; t_1, t_2)\, dx_1\, dx_2 \quad .$$

Note that, in general $E\left[f(x_1)\right]$ is a function of t_1 and $E\left[g(x_1, x_2)\right]$ is a function of t_1 and t_2; however, if the distributions W_{1x} and W_{2x} are stationary, $E\left[f(x_1)\right]$ is time independent and $E\left[g(x_1, x_2)\right]$ depends only on the time difference $t_2 - t_1$.

An average of considerable importance in probability theory is the <u>characteristic function</u>, which, for the second order case, is defined as

$$E\left[\exp(jv_1 x_1 + jv_2 x_2)\right] = M_{2x}(v_1, v_2; t_1, t_2) \quad .$$

If W_{2x} exists then the characteristic function becomes

$$M_{2x}(v_1, v_2; t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(jv_1x_1 + jv_2x_2)\, W_{2x}(x_1, x_2; t_1, t_2)\, dx_1\, dx_2 .$$

Thus the characteristic function is the Fourier transform of the density function W_{2x} (except for the sign of the exponent of e).

The characteristic function has great utility in computing the density function of the sum of two random variables when the individual density functions are known. Consider the first order case of the sum of x_1 and y_1 with $W_{1x}(x_1)$ and $W_{1y}(y_1)$ given. The subscripts will be dropped for convenience (a standard practice) and stationarity of W_{1x} and W_{1y} will be assumed. Let $z = x + y$ and the problem is to find W_{1z}. The characteristic function of z is

$$M_{1z} = E\left[\exp(jvz)\right] = E\left[\exp(jvx + jvy)\right] .$$

If x and y are independent then

$$M_{1z} = M_{1x}\, M_{1y} .$$

The density function of z, W_{1z} can now be computed from

$$W_{1z}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_{1z}(v) \exp(-jvz)\, dv .$$

2. Time Averages

The average over time that is performed on members of the ensemble is defined as

$$A\left[x_\alpha(t)\right] = \lim_{T_1 \to \infty} \frac{1}{2T_1} \int_{-T_1}^{0} x_\alpha(t)\, dt + \lim_{T_2 \to \infty} \frac{1}{2T_2} \int_{0}^{T_2} x_\alpha(t)\, dt$$

where the time average operator is denoted by A. The two integrals usually reduce to the single integral formula

$$A\left[x_\alpha(t)\right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_\alpha(t)\, dt$$

in view of the fact that, when the two integral expression exists, the single integral expression also exists and they are equal. It is possible that the single integral expression may exist when the two integral expression does not for certain functions, and in order to obtain the desired property of time invariance for the linear operator A, the two integral formulation of time average must be used.

The time average of a member function $x_\alpha(t)$ of a random process will, in general, depend on the ensemble variable α. When the time averages of the member functions are independent of α, the random process is said to be uniform. Actually the character of the integral used for ensemble averaging is such that the results obtained are not changed if at most a countable infinity

of member functions do not have the property of uniformity. If the notion of a random process is to be used in the formulation of a mathematical model for a physical system, the assumption of uniformity must generally be made, in that only the member function of the random process is available for experimental observation.

3. Equivalence of Time and Ensemble Averages

Suppose ensemble and time averaging are performed on the function $g(x_1, x_2)$ with reversed order as follows:

$$AE\left[g(x_1, x_2)\right] = EA\left[g(x_1, x_2\right] \quad .$$

Various necessary conditions for the existence of the integral $A\left[g(x_1, x_2\right]$ are given by theorems on ergodic theory. Now if the process is uniform, the relation reduces to

$$AE\left[g(x_1, x_2)\right] = A\left[g(x_1, x_2)\right]$$

since $A\left[g(x_1, x_2)\right]$ is independent of the ensemble variable and, therefore, the ensemble average leaves $A\left[g(x_1, x_2)\right]$ unchanged. Also if $W_{2x}(x_1, x_2)$ is stationary, then

$$E\left[g(x_1, x_2)\right] = A\left[g(x_1, x_2)\right] \quad .$$

This relationship provides the link between ensemble averages computed from the random process model of a physical system, and the measurements that may be made on an actual system using time averages. An excellent and thorough treatment of the general subject of the relationships between ensemble and time averages, which is reviewed only briefly here, may be found in a paper by W. M. Brown (3).

C. CORRELATION FUNCTIONS AND POWER SPECTRA

With the material covered in preceeding sections of this chapter available, a theory for spectral characteristics of random processes can be given without difficulty. The theory presented here will be based on ensemble averages; however, equivalence with the spectral theory obtained for time averages may be quickly demonstrated with the aid of the results of the previous section.

1. <u>Ensemble Correlation Functions</u>

The <u>autocorrelation function</u> for $x_a(t)$ over the ensemble is defined as

$$R_x(t_1, t_2) = E\left[x_1, x_2\right] = \int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} x_1 x_2 \, W_{2x}(x_1, x_2; t_1, t_2)\, dx_1\, dx_2 \ .$$

If W_{2x} is stationary, then R_x is a function of the time difference $t_2 - t_1$ which will be denoted as τ. If a random process has the property that its correlation function depends only on the time difference τ, and if $E\left[x_1^{\,2}\right] = \overline{x_1^{\,2}}$ is independent of time; then the process is called stationary in the wide sense. Cross-correlation functions are defined for two random processes $x_\alpha(t)$ and $y_\alpha(t)$ as

$$R_{xy}(t_1, t_2) = E\left[x_1 y_2\right]$$

however, we shall not have need of functions of this type; consequently the terms autocorrelation and correlation will be synonymous for the remainder of this report.

An important property of correlation functions can be easily derived by observing that the expected value of a non-negative quantity is non-negative. Thus

$$E\left[\left(\frac{x_1}{\sqrt{\overline{x_1^{\,2}}}} \pm \frac{x_2}{\sqrt{\overline{x_2^{\,2}}}}\right)^2\right] \geq 0$$

or

$$E\left[\frac{x_1^{\,2}}{\overline{x_1^{\,2}}}\right] + E\left[\frac{x_2^{\,2}}{\overline{x_2^{\,2}}}\right] \geq \pm E\left[\frac{2\,x_1\,x_2}{\overline{x_1^{\,2}}\;\overline{x_2^{\,2}}}\right] ,$$

thus

$$\sqrt{\overline{x_1^2}\;\overline{x_2^2}} \geq R_v(t_1, t_2)$$

and for stationary random processes

$$R_x(0) \geq |R_x(\tau)|$$

since

$$\overline{x_1^2} = \overline{x_2^2} = R_x(0) \quad .$$

2. Power Density Spectra of Random Processes

The power density spectrum, $S_x(\omega)$, of a random process $x_\alpha(t)$ is defined as the Fourier transform of the correlation function $R_x(\tau)$ where the transform exists. Note that $x_\alpha(t)$ is assumed to be wide-sense stationary. Since a function is required to be absolutely integrable to insure existence of its Fourier transform, unless $R(\tau)$ goes to zero as τ goes to infinity, the power density spectrum will not exist in the strict sense. Then the correlation function is related to a cumulative power spectrum $\mathbf{S}_x(\omega)$ by

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(j\omega\tau)\, d\,\mathbf{S}_x(\omega) \quad .$$

This last result, which is essentially the famous Wiener-Khinchin Theorem, provides the basic procedure for computing power spectra of random waveforms. As in the case of probability distribution and density functions, it is usually possible to introduce delta functions to account for jump discontinuities in the cumulative power spectrum. The appearance of delta functions in power density spectra accounts for power corresponding to discrete frequencies including $\omega = 0$ (or dc). It should be noted that there exist correlation functions for which the cumulative power spectrum has the property that it is continuous everywhere except at a finite set of points; however, its derivative there is zero (the so called singular case). In this situation, the delta function cannot be used; however, such cases are, fortunately, not commonly encountered in engineering practice. The use of power density spectra containing delta functions will be adequate for the power density spectra that will be computed in this report.

It remains to give a justification for the name power density spectrum for the function which has been rather arbitrarily defined as the Fourier transform of an ensemble correlation function. If the inverse Fourier transform is used to express $R_v(\tau)$ in terms of $S_x(\omega)$ then

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \exp(j\omega\tau)\, d\tau \quad .$$

The correlation function evaluated at $\tau = 0$ is simply the mean square value of $x_a(t)$; thus

$$R_x(0) = \overline{x^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, d\omega \quad .$$

Also it can be shown (2) that the correlation function of the output, $y_a(t)$ of a linear time invariant system can be given in terms of the correlation function of the input, $x_a(t)$, by the triple convolution

$$R_y(\tau) = h(t) * h_c(t) * R_x(t) \quad ,$$

where $h(t)$ is the system impulse response and $h_c(t)$ is the complex conjugate of $h(-t)$. In the frequency domain this becomes

$$S_y(\omega) = |H(j\omega)|^2 \, S_x(\omega) \quad ,$$

where $H(j\omega)$ is the system transfer function (Fourier transform of $h(t)$).

If the system characterized by $H(j\omega)$ is assumed to be a narrow band filter with unit response for $\omega_1 \le \omega \le \omega_2$ and zero elsewhere then

$$R_y(0) = \overline{y^2} = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} S_x(\omega)\, d\omega \quad .$$

Now since $\overline{y^2}$ is always positive and since ω_1 and ω_2 can be closely spaced at any point along the frequency axis ω, it follows that $S_x(\omega)$ must be a non-negative function of ω. Also it can be seen that $S_x(\omega)$ is an even function by observing that

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) \exp(-j\omega\tau)\, d\tau = \int_{-\infty}^{\infty} R_x(\tau) \cos \omega\tau\, d\tau$$

since $R_x(\tau)$ is a real, even function; and, therefore,

$$S_x(-\omega) = \int_{-\infty}^{\infty} R_x(\tau) \cos(-\omega\tau)\, d\tau = \int_{-\infty}^{\infty} R_x(\tau) \cos \omega\tau\, d\tau \ .$$

Thus, $S_x(\omega)$ is seen to correspond to the distribution of power with frequency.

A similar theory can be developed for time averages by defining the autocorrelation function over time as $\mathcal{R}_x(\tau) = A\left[x(t)\ x(t+\tau)\right]$ for uniform random processes. The <u>time power density spectrum</u> is

$$\mathcal{S}_x(\omega) = \int_{-\infty}^{\infty} \mathcal{R}_x(\tau) \exp(-j\omega\tau)\, d\tau \quad .$$

If $x_a(t)$ is not stationary in the wide sense then $R_x(t,\tau) = E\left[x_a(t)\ x_a(t+\tau)\right]$ can be averaged over time to eliminate the t

dependence and a power spectrum is computed as

$$\mathcal{S}_x(\omega) = \int_{-\infty}^{\infty} \exp(-j\omega\tau)\, AE\left[x_a(t)\, x_a(t+\tau)\right] d\tau \quad .$$

This power density spectrum is also denoted $\mathcal{S}_x(\omega)$ since

$$AE\left[x_a(t)\, x_a(t+\tau)\right] = A\left[x_a(t)\, x_a(t+\tau)\right] \quad .$$

Now if $x_a(t)$ is wide sense stationary then

$$E\left[x_a(t)\, x_a(t+\tau)\right] = A\left[x_a(t)\, x_a(t+\tau)\right]$$

and

$$\mathcal{S}_x(\omega) = S_x(\omega) \quad .$$

D. THE GAUSSIAN RANDOM PROCESS

The Gaussian random process is of particular importance in the analysis of physical systems because of its frequent occurrence. For example, many noise voltages encountered in electronic systems are characterized by either Gaussian distributions or nonlinear transformations of Gaussian distributions. The wide-spread occurrence of phenomena that have Gaussian properties is predicted by the central limit theorem which states that distribution of the sum of n independent random variables becomes Gaussian

as n tends to infinity. This would serve to explain the fact that the noise voltage of an ohmic resistor is Gaussian in that it is made up of the combination of a large number of impulses due to thermally induced motion of individual electrons.

The first and second order Gaussian density functions and the corresponding characteristic functions will be recorded here for later use. It will be assumed that the random process has a zero mean value, that is $E\left[x_a(t_1)\right] = 0$

$$W_{1x}(x_1, t_1) = \frac{1}{\sqrt{2\pi\,\sigma_1^{\,2}}} \exp\left(-\frac{x_1^{\,2}}{2\sigma_1^{\,2}}\right) \qquad M_{1x}(v_1) = \exp\left(-\frac{v_1^{\,2}\,\sigma_1^{\,2}}{2}\right)$$

$$W_{2x}(x_1, x_2; t_1, t_2) = \frac{1}{2\pi\left[\sigma_1^{\,2}\,\sigma_2^{\,2} - R_x(t_1, t_2)\right]^{\frac{1}{2}}}$$

$$\exp\left\{-\frac{\sigma_1^{\,2}\,x_1^{\,2} - 2R_x(t_1, t_2)\,x_1x_2 + \sigma_2^{\,2}\,x_2^{\,2}}{2\left[\sigma_1^{\,2}\,\sigma_2^{\,2} - R_x(t_1, t_2)\right]}\right\}$$

$$M_{2x}(v_1, v_2; t_1, t_2) = \exp\left\{-\frac{1}{2}\left[\sigma_1^{\,2} v_1^{\,2} + 2R_x(t_1, t_2)\,v_1v_2 + \sigma_2^{\,2}\,v_2^{\,2}\right]\right\}.$$

An important property of Gaussian distributed quantities is the fact that if the input to a linear system is Gaussian, then the output is Gaussian. In other words linear transformations of Gaussian noise give Gaussian noise. (See Reference 4 for details.) It is true, however, that a linear transformation may well result in Gaussian distributions that are non-stationary.

III. FORMULATION OF THE RANDOM MODULATED WAVE PROBLEM

A. THE ENSEMBLE CORRELATION FUNCTION OF A GENERAL RANDOM MODULATED WAVE

A general expression for the random time function whose spectral properties will be investigated is as follows:

$$v(t) = C(t) \cos \left[\omega_c t + \phi(t) + \theta \right] \quad .$$

Here, specifically, $v(t)$ represents the instantaneous voltage output of a sinusoidal oscillator, of radian frequency ω_c, whose amplitude is modulated by $C(t)$ and whose instantaneous radian frequency is given by $\omega_c + \dot{\phi}(t)$ with θ being a fixed arbitary phase angle. Now if $C(t)$ and $\phi(t)$ are random variables, then $v(t)$ is a random variable. The ensemble parameter, α, will not be indicated explicitly; however, it is to understood that $v(t)$, $C(t)$, and $\phi(t)$ represent member functions of random processes.

The correlation function of $v(t)$ will now be formulated; however, the question as to wide sense stationarity of $v(t)$ will be left open for the moment, and R_v will be indicated as a function of both t_1 and t_2, or equivalently $t = t_1$ and $\tau = t_2 - t_1$.

$$R_v(t,\tau) = E\left[v(t)\, v(t+\tau)\right]$$

$$= E\left\{C(t)\, C(t+\tau) \cos\left[\omega_c t+\phi(t)+\theta\right] \cos\left[\omega_c t+\omega_c \tau+\phi(t+\tau)+\theta\right]\right\}.$$

Using the trigonometric identity for the product of two cosine functions along with the linearity property of the expected value operator E gives

$$R_v(t,\tau) = E\left\{\frac{1}{2}\, C(t)\, C(t+\tau) \cos\left[2\omega_c t + \omega_c \tau + \phi(t+\tau) + \phi(t) + 2\theta\right]\right\}$$

$$+ E\left\{\frac{1}{2}\, C(t)\, C(t+\tau) \cos\left[\omega_c \tau + \phi(t+\tau) - \phi(t)\right]\right\} .$$

The arbitrary phase angle θ is assumed to be uniformly distributed over the interval $0 \leq \theta \leq 2\pi$, that is

$$W_{1\theta}(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{for all other } \theta \end{cases} .$$

The required joint probability density function for the expected value integral would have the form $W_{2C}\, W_{2\phi}\, W_{1\theta}$ on the assumption that the random variables C, ϕ, and θ are independent. In carrying out the indicated expected value operations to obtain the correlation function $R_v(t,\tau)$, the term which involves the phase

angle θ will vanish. This can be seen by noting that the expression

$$E\left\{\frac{1}{2} C(t)\, C(t+\tau) \cos\left[2\omega_c t + \omega_c \tau + \phi(t+\tau) + \phi(t) + 2\theta\right]\right\}$$

takes the form

$$\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(t,\tau) \int_0^{2\pi} \cos\left[\beta(t,\tau)+2\theta\right] W_{1\theta}(\theta)\, d\theta \;\; W_{2C} W_{2\phi}$$

$$dC_1\, dC_2\, d\phi_1\, d\phi_2$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(\beta + 2\theta)\, d\theta = 0 \quad .$$

Thus the correlation function of $v(t)$ reduces to

$$R_v(t,\tau) = E\left\{\frac{1}{2} C(t)\, C(t+\tau) \cos\left[\omega_c \tau + \phi(t+\tau) - \phi(t)\right]\right\} .$$

In order to proceed further the functions $C(t)$ and $\phi(t)$ must e specified. The following cases will be considered in detail in the chapters to follow:

(1) Case I. FM by band-pass noise only. $C(t)$ is a constant C_0, and $\phi(t)$ is obtained by integrating band-pass noise, that is

$$\phi(t) = D \int_{-\infty}^{t} v_n(t')\, dt' \quad ,$$

where D is the frequency sensitivity of the sinusoidal oscillator in radians/volt and $v_n(t)$ is a stationary Gaussian noise voltage with zero mean.

(2) Case II. FM by band-pass noise combined with a sine wave. This is the same as Case I except that $\phi(t)$ is now

$$\phi(t) = D \int_{-\infty}^{t} \left[v_n(t') - E_m \sin(\omega_m t' + \alpha) \right] dt' \quad ,$$

where E_m and ω_m are the peak value and radian frequency of a sine wave respectively, and α is a uniformly distributed random phase angle. The band-pass noise spectrum will be centered at ω_m.

(3) Case III. FM by band-pass noise with simultaneous AM by low-pass noise. The function $\phi(t)$ is the same as in Case I but $C(t)$ is now $C(t) = C_0 + v_a(t)$ where $v_a(t)$ is a stationary Gaussian noise voltage with zero mean and having a low-pass power density spectrum. Also $v_n(t)$ and $v_a(t)$ are assumed to be independent.

IV. FM BY BAND-PASS NOISE

The development of this chapter for the case of FM by band-pass noise represents the principal contribution of this report. A method of approximation with which useful information concerning the shape of the power density spectrum as a function of the essential modulation parameters will be explored in detail.

A. SPECIFICATION OF THE CORRELATION FUNCTION IN TERMS OF THE MODULATING NOISE SPECTRUM

The correlation function for FM by band-pass noise was derived in Chapter III as

$$R_v(t,\tau) = \frac{C_0^{\,2}}{2} E\left\{\cos\left[\omega_c t + \phi(t+\tau) - \phi(t)\right]\right\},$$

with

$$\phi(t) = D \int_{-\infty}^{t} v_n(t')\,dt' \quad .$$

If the power density spectrum $S_{v_n}(\omega)$, of the modulating noise is specified, then the correlation function can be evaluated as will be shown subsequently. Also the issue as to the wide sense stationarity of v(t), or equivalently the question of time indepen-dependence of R_v, will be settled.

The correlation function, $R_v(t,\tau)$, can be written in the following form where the equivalent time variables, $t_1 = t$ and $t_2 = t+\tau$, have been used in place of t and τ, so that $\phi(t+\tau) = \phi(t_2) = \phi_2$ and $\phi_1 = \phi(t_1) = \phi(t)$.

$$R_v(t,\tau) = \frac{C_0^2}{2} \operatorname{Re}\left\{\exp(j\omega_c\tau)\, E\left[\exp(j\phi_2 - j\phi_1)\right]\right\} \quad .$$

The symbol Re denotes "real part of". From the definition of the second order characteristic function of ϕ, this is seen to be

$$R_v(t_1,t_2) = \frac{C_0^2}{2} \operatorname{Re}\left[\exp(j\omega_c\tau)\, M_{2\phi}(-1,1;t_1,t_2)\right] \quad .$$

Since ϕ is a linear transformation of a Gaussian process, it too will be Gaussian, but not necessarily stationary; therefore

$$M_{2\phi}(-1,1;t_1,t_2) = \exp\left\{-\frac{1}{2}\left[\overline{\phi_1^2} - 2R_\phi(t_1,t_2) + \overline{\phi_2^2}\right]\right\} .$$

The terms $\overline{\phi_1^2}$, $\overline{\phi_2^2}$, and $R_\phi(t_1,t_2) = \overline{\phi_1\phi_2}$ will now be compute from the defining relation for ϕ, that is

$$\phi(t) = D\int_{-\infty}^{t} v_n(t')\,dt' \quad .$$

The mean square value of ϕ can be written as

$$\overline{\phi_i^{\,2}} = E\left\{\left[D \int_{-\infty}^{t_i} v_n(x)\, dx\right]\left[D \int_{-\infty}^{t_i} v_n(y)\, dy\right]\right\}$$

$$= D^2 \int_{-\infty}^{t_i} \int_{-\infty}^{t_i} E\left[v_n(x)\, v_n(y)\right] dy\, dx$$

for $i = 1, 2$.

Thus

$$\overline{\phi_i^{\,2}} = D^2 \int_{-\infty}^{t_i} \int_{-\infty}^{t_i} R_{v_n}(y-x)\, dy\, dx$$

since v_n is stationary. Similarly

$$R_\phi(t, \tau) = \overline{\phi_1 \phi_2} = D^2 \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} R_{v_n}(y-x)\, dy\, dx \quad .$$

Let $y-x = u$ and $y = v$. This change of variables maps the x, y plane into the u, v plane as shown in Figure 1.

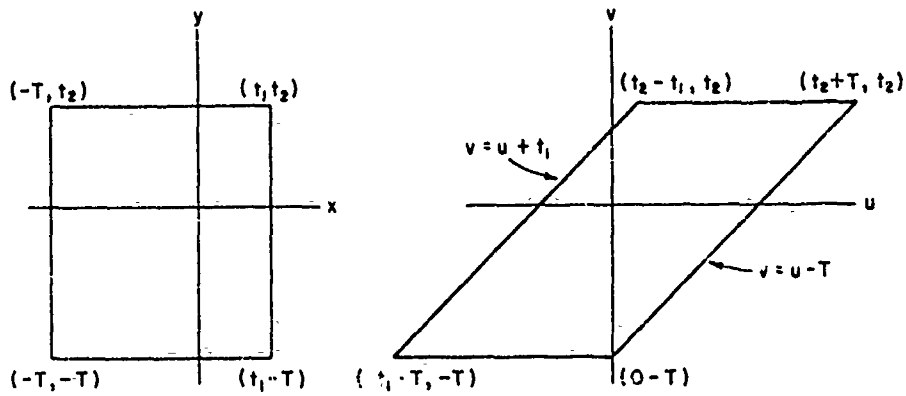

FIGURE 1 MAPPING OF x, y TO u, v PLANE

The integral for $R_\phi(t,\tau)$ is obtained for $T \to \infty$. In terms of the new variables u and v, the integral becomes

$$\overline{\phi_1\phi_2} = D^2\left\{\int_{-\infty}^{0} R_{v_n}(u)\left[\int_{-\infty}^{u+t_1} dv\right]du + \int_{0}^{\tau} R_{v_n}(u)\left[\int_{-\infty}^{u+t_1} dv\right]du\right.$$

$$\left. + \int_{\tau}^{\infty} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv\right]du\right\}.$$

Similarly $\overline{\phi_1^2}$ and $\overline{\phi_2^2}$ become

$$\overline{\phi_1^2} = D^2 \left\{ \int_{-\infty}^{0} R_{v_n}(u) \left[\int_{-\infty}^{u+t_1} dv \right] du + \int_{0}^{\infty} R_{v_n}(u) \left[\int_{-\infty}^{t_1} dv \right] du \right\}$$

and

$$\overline{\phi_2^2} = D^2 \left\{ \int_{-\infty}^{0} R_{v_n}(u) \left[\int_{-\infty}^{u+t_2} dv \right] du + \int_{0}^{\infty} R_{v_n}(u) \left[\int_{-\infty}^{t_2} dv \right] du \right\}.$$

The quantity $\left[\overline{\phi_1^2} + \overline{\phi_2^2} - 2\, \overline{\phi_1 \phi_2} \right]$ can now be evaluated.

$$\frac{\overline{\phi_1^2} + \overline{\phi_2^2}}{D^2} = \int_{-\infty}^{0} R_{v_n}(u) \left[\int_{-\infty}^{u+t_1} dv \right] du + \int_{0}^{\infty} R_{v_n}(u) \left[\int_{-\infty}^{t_1} dv \right] du$$

$$+ \int_{-\infty}^{0} R_{v_n}(u) \left[\int_{-\infty}^{u+t_2} dv \right] du + \int_{0}^{\infty} R_{v_n}(u) \left[\int_{-\infty}^{t_2} dv \right] du$$

$$= \int_{0}^{\infty} R_{v_n}(u) \left[\int_{-\infty}^{-u+t_1} dv + \int_{-\infty}^{-u+t_2} dv + \int_{-\infty}^{t_1} dv + \int_{-\infty}^{t_2} dv \right] du ,$$

where use has been made of the fact that $R_{v_n}(u)$ is an even function.

In a similar manner $\dfrac{\overline{\phi_1\phi_2}}{D^2}$ becomes

$$\frac{\overline{\phi_1\phi_2}}{D^2} = \int_0^{\infty} R_{v_n}(u)\left[\int_{-\infty}^{-u+t_1} dv\right]du + \int_0^{\tau} R_{v_n}(u)\left[\int_{-\infty}^{u+t_1} dv\right]du$$

$$+ \int_{\tau}^{\infty} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv\right]du \quad .$$

Thus

$$\frac{\overline{\phi_1^2}+\overline{\phi_2^2}-2\,\overline{\phi_1\phi_2}}{D^2} = \int_0^{\infty} R_{v_n}(u)\left[\int_{-u+t_2}^{-u+t_2} dv - \int_{t_1}^{t_2} dv + 2\int_{-\infty}^{t_2} dv\right]du$$

$$-2\int_0^{\tau} R_{v_n}(u)\left[\int_{-\infty}^{u+t_1} dv\right]du - 2\int_{\tau}^{\infty} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv\right]du$$

$$= 2\int_0^{\infty} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv\right]du - 2\int_0^{\tau} R_{v_n}(u)\left[\int_{-\infty}^{u+t_1} dv\right]du$$

$$-2\int_0^{\infty} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv\right]du + 2\int_0^{\tau} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv\right]du$$

$$= 2\int_0^{\tau} R_{v_n}(u)\left[\int_{-\infty}^{t_2} dv - \int_{-\infty}^{u+t_1} dv\right]du$$

$$= 2\int_0^{\tau} R_{v_n}(u)\left[\int_{u+t_1}^{t_2} dv\right]du$$

$$= 2\int_0^{\tau} (\tau-u)\, R_{v_n}(u)\, du \quad .$$

At this point the correlation function $R_{v_n}(u)$ is expressed in terms of the corresponding power density spectrum $S_{v_n}(\omega)$ to get

$$\frac{\overline{\phi_1^2} + \overline{\phi_2^2} - 2\,\overline{\phi_1\phi_2}}{D^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} S_{v_n}(\omega) \int_0^{\tau} (\tau - u) \cos \omega u \, du \, d\omega \quad ,$$

where the Fourier cosine transform expression for $S_{v_n}(\omega)$ is used, that is

$$R_{v_n}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{v_n}(\omega) \cos \omega\tau \, d\omega \quad .$$

The integral over u is easily evaluated to get finally

$$\overline{\phi_1^2} + \overline{\phi_2^2} - 2\,\overline{\phi_1\phi_2} = \frac{D^2}{\pi} \int_{-\infty}^{\infty} S_{v_n}(\omega) \frac{(1 - \cos \omega\tau)}{\omega^2} \, d\omega \quad .$$

Thus the correlation function for $v(t)$ becomes

$$R_v(\tau) = \frac{C_0^{\,2}}{2} \cos \omega_c \tau \exp\left[- \frac{D^2}{2\pi} \int_{-\infty}^{\infty} S_{v_n}(\omega) \frac{(1 - \cos \omega\tau)}{\omega^2} \, d\omega \right].$$

Note that R_v has been shown to be independent of time.

B. THE POWER DENSITY SPECTRUM OF WHITE NOISE PROCESSED BY A SINGLE STAGE LRC FILTER

The modulating voltage $v_n(t)$ is obtained by filtering "white" Gaussian noise by a single stage LRC filter such as that shown in Figure 2 below.

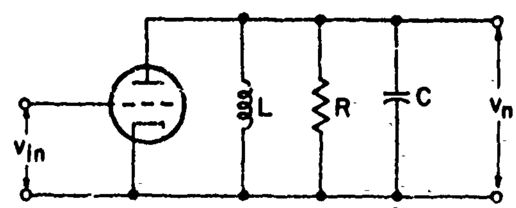

FIGURE 2 SINGLE STAGE LRC FILTER

The transfer function of this filter is

$$H(s) = -\frac{g_m}{C} \frac{s}{s^2 + \frac{1}{RC} s + \frac{1}{LC}}$$

where g_m is the transconductance of the tube. Let

$$\omega_1 = \frac{1}{RC} \qquad \text{and} \qquad \omega_0{}^2 = \frac{1}{LC} .$$

The amplitude frequency response squared becomes

$$|H(j\omega)|^2 = \frac{H_0^2\,\omega^2}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\omega^2} \quad \text{with} \quad H_0^2 = \frac{g_m^2}{C^2} \quad .$$

If the power density spectrum of $v_{in}(t)$ is white, that is, $S_{v_{in}}(\omega) = N$ for all ω, then the power density spectrum of the output $v_n(t)$ is

$$S_{v_n}(\omega) = \frac{H_0^2 N\,\omega^2}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\,\omega^2} \quad .$$

The mean square value of $v_n(t)$ is given by

$$\overline{v_n^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{v_n}(\omega)\,d\omega = \frac{H_0^2 N}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\,\omega^2}\,d\omega \ .$$

Evaluation of the integral[3] yields $\frac{\pi}{\omega_1}$. Thus

$$\overline{v_n^2(t)} = \frac{H_0^2 N}{2\,\omega_1} \quad .$$

---

[3]See Appendix A.

In applying the modulating voltage, $v_n(t)$, we will assume that N is adjusted in order to maintain $\overline{v_n^2(t)}$ constant whenever ω_1 is changed. Thus $S_{v_n}(\omega)$ becomes

$$S_{v_n}(\omega) = \frac{2\,\overline{v_n^2}\,\omega_1\,\omega^2}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\,\omega^2} \,.$$

Plots of this function for values of $Q = \frac{\omega_1}{\omega_0}$ of 10 and 20 are given in Figure 3.

C. THE CORRELATION FUNCTION OF FM BY BAND-PASS NOISE

The equation, derived in Section IV-A, which gives $R_v(\tau)$ in terms of $S_{v_n}(\omega)$ can now be evaluated by use of the expression for $S_{v_n}(\omega)$ of Section IV-B. Thus

$$R_v(\tau) = \frac{C_0^2}{2}\cos\omega_c\tau\,\exp\left[-\frac{D^2\,\overline{v_n^2}\,\omega_1}{\pi}\int_{-\infty}^{\infty}\frac{(1-\cos\omega\tau)}{(\omega_0^2-\omega^2)^2+\omega_1^2\,\omega^2}\,d\omega\right].$$

Upon evaluation of the integral[4] over ω the following expression for $R_v(\tau)$ is obtained:

$$R_v(\tau) = \frac{C_0^2}{2}\cos\omega_c\tau\,\exp\left\langle -\frac{D^2\,\overline{v_n^2}}{\omega_0^2}\left\{1-\exp\left(-\frac{\omega_1|\tau|}{2}\right)\left[\cos\sqrt{\omega_0^2-\frac{\omega_1^2}{4}}\,|\tau| + \frac{\omega_1}{2\sqrt{\omega_0^2-\frac{\omega_1^2}{4}}}\sin\sqrt{\omega_0^2-\frac{\omega_1^2}{4}}\,|\tau|\right]\right\}\right\rangle \,.$$

[4] See Appendix B.

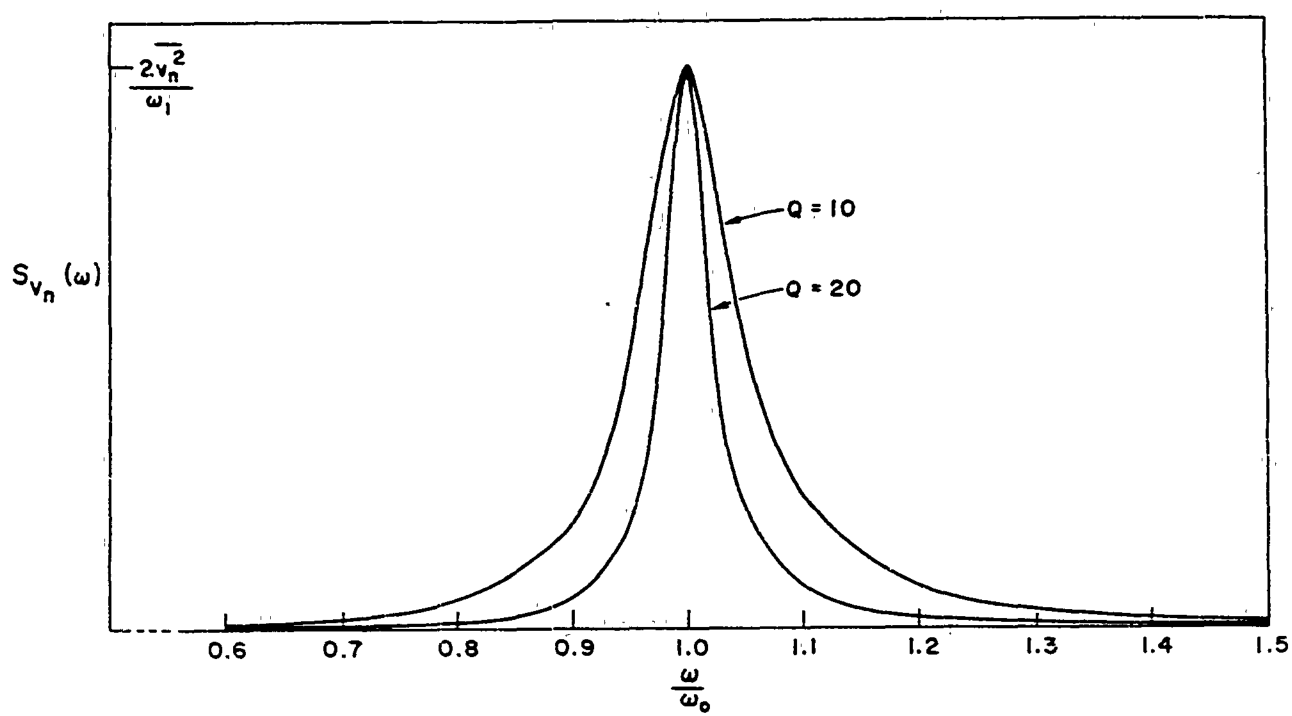

FIGURE BAND-PASS MODULATING NOISE SPECTRA

Let $m^2 = \frac{D^2 \overline{v_n^2}}{\omega_0^2}$. The quantity m is the modulation index in that it is the ratio of the rms frequency deviation to the center frequency of the modulation. With the use of the definition of $Q = \frac{\omega_0}{\omega_1}$ the correlation function $R_v(\tau)$ becomes

$$R_v(\tau) = \frac{C_0^2}{2} \cos \omega_c \tau \exp\left\langle -m^2 \left\{ 1 - \exp\left(- \frac{\omega_0 |\tau|}{2Q}\right) \left[\cos b\, \omega_0 |\tau| + \frac{1}{2Qb} \sin b\, \omega_0 |\tau| \right] \right\} \right\rangle ,$$

where

$$b = \sqrt{1 - \frac{1}{4Q^2}} \quad .$$

Before attempting to evaluate this expression it will be helpful to consider limiting cases of the spectrum $S_v(\omega)$ as m becomes very small or very large.

D. LIMITING CASES FOR FM BY BAND-PASS NOISE

1. Small Modulation Index

Starting with the expression for the correlation function of v(t) in terms of $S_{v_n}(\omega)$, the power density spectrum of the modulating noise, we can obtain an expansion of the spectrum, $S_v(\omega)$, that is appropriate for small values of m.

$$R_v(\tau) = \frac{C_0^2}{2} \cos \omega_c \tau \exp\left[-\frac{D^2}{2\pi} \int_{-\infty}^{\infty} S_{v_n}(\omega) \frac{(1-\cos \omega\tau)}{\omega^2} d\omega\right].$$

Let $\mu^2 = \frac{m_0^2}{v_n^2}$, then $D^2 = m^2\mu^2$ and

$$R_v(\tau) = \frac{C_0^2}{2} \cos \omega_c \tau \exp\left[-\frac{m^2\mu^2}{2\pi} \int_{-\infty}^{\infty} S_{v_n}(\omega) \frac{(1-\cos \omega\tau)}{\omega^2} d\omega\right]$$

$$= \frac{C_0^2}{2} \cos \omega_c \tau \exp\left[-\frac{m^2\mu^2}{2\pi} \int_{-\infty}^{\infty} \frac{S_{v_n}(\omega)}{\omega^2}\right] \times$$

$$\exp\left[\frac{m^2\mu^2}{2\pi} \int_{-\infty}^{\infty} \frac{S_{v_n}(\omega) \cos \omega\tau}{\omega^2} d\omega\right].$$

The second exponential is now expanded in its power series to give

$$R_v(\tau) = \frac{C_0^2}{2} \cos \omega_c \tau \exp\left(-\frac{m^2\mu^2 I}{2\pi}\right)\left\{1 + \frac{m^2\mu^2}{2\pi} \int_{-\infty}^{\infty} \frac{S_{v_n}(\omega) \cos \omega\tau}{\omega^2} d\omega + \frac{m^4\mu^4}{4\pi} \times\right.$$

$$\left.\left[\int_{-\infty}^{\infty} \frac{S_{v_n}(\omega) \cos \omega\tau}{\omega^2} d\omega\right]^2 + \cdots\right\}$$

Notice that in order for this expansion to be valid, the integral $I = \int_{-\infty}^{\infty} \frac{S_{v_n}(\omega)}{\omega^2} d\omega$ must converge. This condition is clearly met

by the $S_{v_n}(\omega)$ under consideration. $S_v(\omega)$ is now obtained by taking the Fourier transform of $R_v(\tau)$

$$S_v(\omega) = \frac{C_0}{4} \exp\left(-\frac{m^2\mu^2 I}{2\pi}\right)\left\{2\pi\left[\delta(\omega+\omega_c)+\delta(\omega-\omega_c)\right]\right.$$

$$\left. + \frac{m^2\mu^2}{2\pi}\left[\frac{S_{v_n}(\omega+\omega_c)}{(\omega+\omega_c)^2}+\frac{S_{v_n}(\omega-\omega_c)}{(\omega-\omega_c)^2}\right]+\cdots\right\}$$

For small m the higher ordered terms (m^4, m^6, etc.) can be neglected; thus the resulting power density spectrum of $v(t)$ is seen to contain a carrier spike at ω_c and sidebands about the carrier which are simply related to the modulating noise power spectrum S_{v_n}. The case for small m but with $S_{v_n}(\omega)$ such that the integral I does not converge can also be treated. See Reference 5.

2. Large Modulation Index

In the final expression for $R_v(\tau)$ in Section IV-C let $a = \frac{1}{2Q}$ and $\omega_0\tau = t$. Then

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^2}{2}\cos\frac{\omega_c t}{\omega_0}\exp\left\{-m^2\left[1-\exp(-a|t|)(\cos bt+\frac{a}{b}\sin b|t|)\right]\right\}.$$

It is clear that for large values of m the correlation function becomes very small as $|t|$ increases. Therefore a good approximation of the correlation function will be obtained if the terms in the bracket are expanded in their power series and only the lower order terms are retained. Let the bracketed term equal f(t), that is,

$$f(t) = 1 - \exp(-a|t|)\left(\cos bt + \frac{a}{b}\sin b|t|\right) \quad .$$

Expanding the exponential and the cosine and sine in their power series gives

$$f(t) = 1 - \left(1 - a|t| + \frac{a^2t^2}{2!} - \cdots\right)\left(1 - \frac{b^2t^2}{2!} + \cdots + \frac{a}{b}b|t| - \frac{a}{b}\frac{b^3|t^3|}{3!} + \cdots\right)$$

$$= 1 - \left[1 - \frac{1}{2}(a^2 - b^2)\,t^2 + \cdots\right]$$

$$= \frac{1}{2}(a^2 - b^2)\,t^2 + \cdots$$

But $a^2 = \frac{1}{4Q^2}$ and $b^2 = 1 - \frac{1}{4Q^2}$; thus

$$f(t) = \frac{1}{2}t^2$$

when terms of higher order than t^2 are neglected.

The correlation function $R_v(\tau)$ now becomes

$$R_v(\tau) = \frac{C_0^2}{2} \cos \omega_c \tau \exp\left(- \frac{D^2 \overline{v_n^2}}{\omega_0^2} \frac{1}{2} \omega_0^2 \tau^2\right) \quad ,$$

or, if $(\Delta\omega)^2 = D^2 \overline{v_n^2}$, where $\Delta\omega$ is the mean square deviation of the carrier ω_c,

$$R_v(\tau) = \frac{C_0^2}{2} \cos \omega_c \tau \quad \exp\left[- \frac{(\Delta\omega)^2}{2} \quad \tau^2\right] \; .$$

The power density spectrum is now obtained by use of the Fourier cosine transform

$$S_v(\omega) = \frac{C_0^2}{2} \int_{-\infty}^{\infty} \exp\left[- \frac{(\Delta\omega)^2 \tau^2}{2}\right] \cos \omega_c \tau \cos \omega\tau \, d\tau \; .$$

Using the trigonometric identity for the product of cosines gives

$$S_v(\omega) = \frac{C_0^2}{4} \int_{-\infty}^{\infty} \exp\left[- \frac{(\Delta\omega)^2 \tau^2}{2}\right] \cos (\omega + \omega_c)\tau \; d\tau$$

$$+ \frac{C_0^2}{4} \int_{-\infty}^{\infty} \exp\left[- \frac{(\Delta\omega^2) \tau^2}{2}\right] \cos (\omega - \omega_c)\tau \; d\tau$$

$$= S_v^-(\omega) + S_v^+(\omega) \quad .$$

If $\Delta\omega \ll \omega_c$, then there is negligible contribution to the spectrum $S_v(\omega)$ by S_v^- for $\omega > 0$, and similarly $S_v^-(\omega)$ is very nearly equal to $S_v(\omega)$ for $\omega < 0$. It suffices, then, to consider only $S_v^+(\omega)$. From Pierce's Table of Integrals No. 508 (6)

$$S_v^+(\omega) = \frac{C_0^{\,2}}{4} \, \frac{\sqrt{2\pi}}{\Delta\omega} \exp\left[-\frac{(\omega-\omega_c)^2}{2(\Delta\omega)^2}\right] .$$

The validity of this result is easily checked by noting that $S_v(\omega) = S^+(\omega) + S^-(\omega) = S^+(\omega) + S^+(-\omega)$ and

$$\overline{v_n^{\,2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_v(\omega)\, d\omega$$

$$= \frac{C_0^{\,2}}{2} \, \frac{1}{\sqrt{2\pi}\,\Delta\omega} \int_{-\infty}^{\infty} \exp\left[-\frac{(\omega-\omega_c)^2}{2(\Delta\omega)^2}\right] d\omega = \frac{C_0^{\,2}}{2} \quad .$$

Thus the power in the carrier, ω_c, is effectively redistributed into a continuous spectrum by the process of frequency modulation. Notice that the spectrum of the modulated wave $v(t)$ has become Gaussian as the modulation index $m \to \infty$ and thus, has the same form as the probability density function of the modulating voltage $v_n(t)$. Middleton (5) has shown that, under quite general conditions, the power density spectrum of a frequency modulated wave tends to assume the shape of the first order probability density function of the modulating voltage when the modulation index goes to infinity[5].

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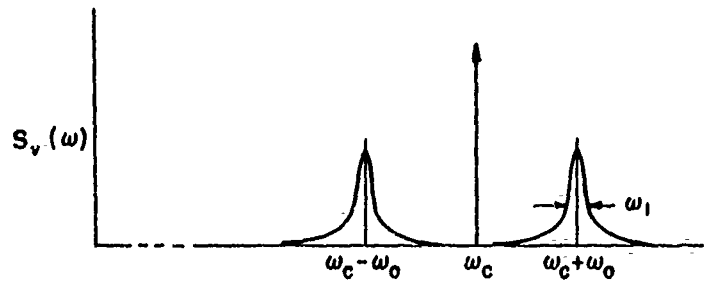
[5]See also References 7, 8, and 9.

Now that the cases of both small and large modulation indices have been examined, it is clear that the power density spectrum of FM by band-pass noise undergoes a rather striking transition as the modulation index is increased as is shown in Figure 4. The question to be considered in the remainder of this chapter is the spectral shape that is obtained for intermediate values of the modulation index, m, as a function of the bandwidth of the modulating noise, or equivalently, the filter Q. Ultimately it is desired to obtain a measure of the deviation from the limiting Gaussian spectrum that will result for various values of m and Q.

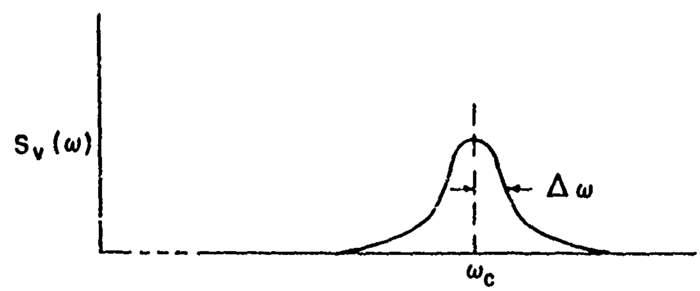
E. DIGITAL COMPUTATION OF FM BY BAND-PASS NOISE POWER DENSITY SPECTRA

In order to provide a basis for comparison with power density spectra that are obtained by use of an approximation technique, the Fourier transform of $R_v(\tau)$ was performed by digital computation for selected values of m and Q. Returning to a form given previously on page 48,

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^2}{2} \exp\left[-m^2 f(t)\right] \cos \frac{\omega_c t}{\omega_0} = \frac{C_0^2}{4} \exp\left[-m^2 f(t)\right]\left[\exp\left(j\frac{\omega_c t}{\omega_0}\right) + \exp\left(-j\frac{\omega_c t}{\omega_0}\right)\right]$$



a. LOW MODULATION INDEX



b. HIGH MODULATION INDEX

FIGURE 4 LIMITING CASES OF FM BY BAND-PASS NOISE SPECTRA

where

$$f(t) = 1 - \exp(-a|t|)(\cos bt + \frac{a}{b} \sin b|t|) \quad ,$$

we have that

$$S_v(\omega) = \int_{-\infty}^{\infty} R_v(\tau) \exp(-j\omega\tau)\, d\tau = \int_{-\infty}^{\infty} R_v(\frac{t}{\omega_0}) \exp(-\frac{j\omega t}{\omega_0}) \frac{dt}{\omega_0}$$

$$= \frac{C_0^2}{4\omega_0} \int_{-\infty}^{\infty} \exp\left[-m^2 f(t)\right] \exp\left[-j\frac{(\omega+\omega_c)t}{\omega_0}\right] dt + \frac{C_0^2}{4\omega_0} \int_{-\infty}^{\infty}$$

$$\exp\left[-m^2 f(t)\right] \exp\left[-j\frac{(\omega-\omega_c)t}{\omega_0}\right] dt \quad .$$

Again if the assumption that $\omega_c >> \Delta\omega$ is made it will suffice to consider only one of these integrals. The second integral, which will give the behavior of the spectrum around $\omega = \omega_c$ and $\beta = \frac{\omega-\omega_c}{\omega_0}$, becomes

$$S_v^+(\beta) = \frac{C_0^2}{4\omega_0} \int_{-\infty}^{\infty} \exp\left[-m^2 f(t)\right] \exp(-j\beta t)\, dt \quad .$$

Note that this integral does not exist in the strict sense because

$$\lim_{t \to 0} \exp\left[-m^2 f(t)\right] = \exp(-m^2) \quad ;$$

however $\exp(-m^2)$ can be subtracted and added to give an expression of the form

$$S_v^+(\beta) = \frac{C_0^{\ 2}}{4\omega_0} \int_{-\infty}^{\infty} \exp\left[-m^2 f(t) - m^2\right] \exp(-j\beta t)\, dt + \frac{C_0^{\ 2}}{4\omega_0} \int_{-\infty}^{\infty} \exp(-m^2) \exp(-j\beta t)\, dt$$

$$= \frac{C_0^{\ 2}}{2\omega_0} \int_0^{\infty} \exp\left[-m^2(t) - m^2\right] \cos \beta t\, dt + \frac{C_0^{\ 2}\pi}{2\omega_0} \exp(-m^2)\, \delta(\beta) \ .$$

The delta function term represents residual power at the carrier frequency. Computation of the first integral was performed for $m = 4$, with $Q = 10,\ 20$. The results of the computation are plotted in Figure 5.

F. A TECHNIQUE FOR COMPUTING FM BY BAND-PASS NOISE POWER DENSITY SPECTRA

It was shown in Section IV-D-2 that the power density spectrum of FM by band-pass Gaussian noise assumes a Gaussian-like shape as the modulation index becomes very large. In this section an approximation technique is introduced which will make possible a determination of the actual value of the modulation index that is required to produce a power spectrum that will deviate from the limiting Gaussian spectrum by a specified amount. As one would expect, the required modulation index will be dependent on the bandwidth of the modulating noise (or equivalently the Q of the modulating noise filter).

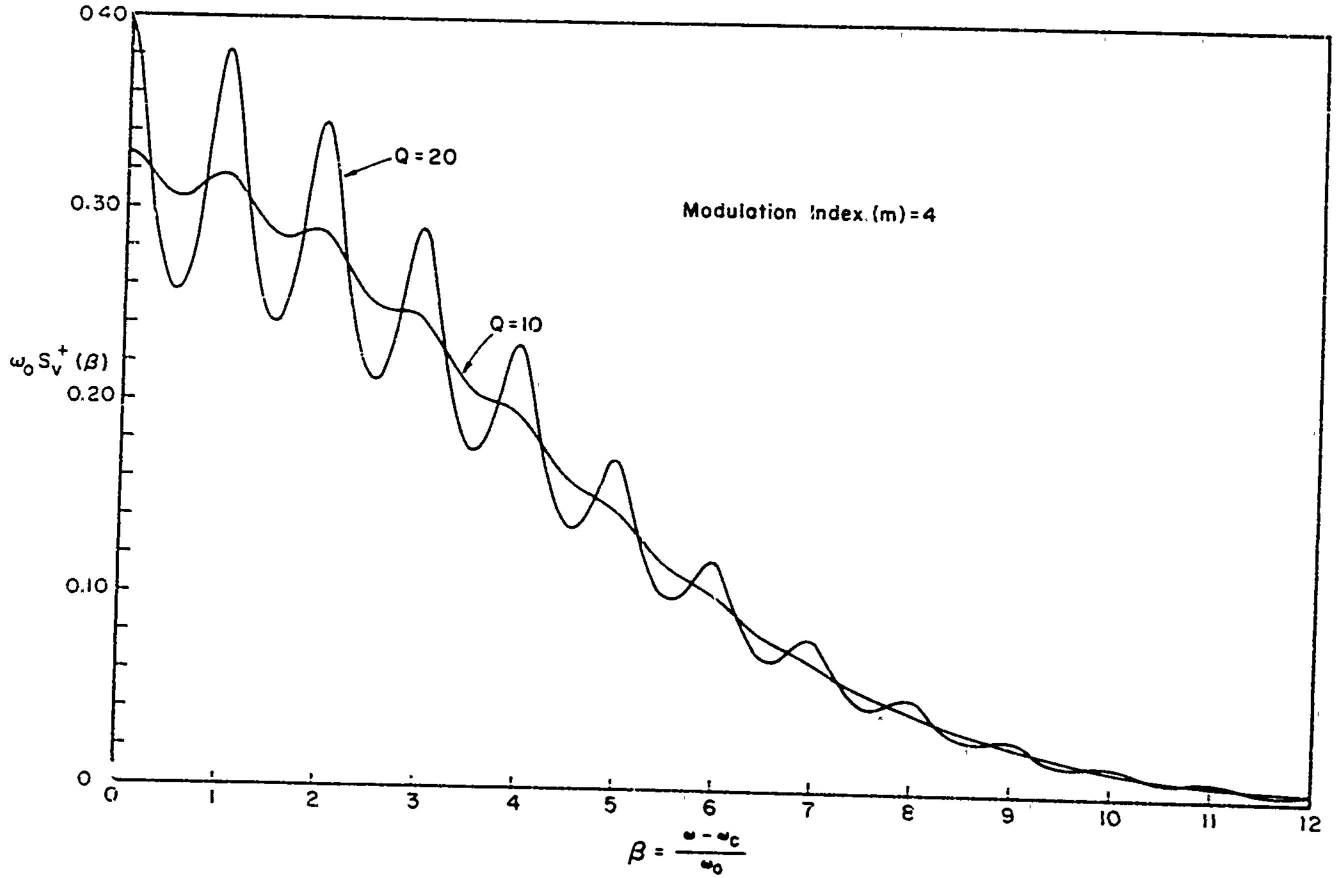


FIGURE 5 FM BY BAND-PASS NOISE POWER DENSITY SPECTRA (BY DIGITAL COMPUTATION)

The starting point in this analysis is again the correlation function of v(t) which is reproduced below for convenience:

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^{\,2}}{2} \exp\left[-m^2 f(t)\right] \cos \frac{\omega_c t}{\omega_0} \quad ,$$

where

$$f(t) = 1 - \exp(-a|t|)(\cos b|t| + \frac{a}{b} \sin b|t|) \; ,$$

with

$$a = \frac{1}{2Q} \; , \quad b = \sqrt{1 - \frac{1}{4Q^2}} \; , \quad \text{and} \quad t = \omega_0 \tau \; .$$

It can be readily seen that the envelope of $R_v(\frac{t}{\omega_0})$, that is, $\frac{C_0^{\,2}}{2} \exp\left[-m^2 f(t)\right]$, which is an even function, has successive maxima at the points $|t| = \frac{2n\pi}{b}$ and minima at $|t| = \frac{(2n+1)\pi}{b}$ with $n = 0, 1, 2, \cdots$. Attention will be concentrated on the behavior of $\exp\left[-m^2 f(t)\right]$ in the vicinity of the maximum points. The importance of these points in determining the shape of the resulting power spectrum becomes obvious when one observes that the envelope function decreases rapidly on either side of the maximum points even for relatively modest values of m, for example on the order of four.

To determine the behavior of the envelope near the critical points for $t > 0$ let $x = t - \frac{2n\pi}{b}$; then

$$\exp\left[-m^2 f(t)\right] = \exp\left\langle\left[-m^2\left\{1 - \exp\left(\frac{2n\pi a}{b}\right)\exp(-a|x|)\right.\right.\right.$$

$$\left.\left.\left.\left[\cos(b|x| + 2n\pi) + \frac{b}{a}\sin(b|x| + 2n\pi)\right]\right\}\right]\right\rangle$$

$$= \exp\left\{-m^2\left[1 - \exp\left(\frac{2n\pi a}{b}\right)\exp(-a|x|)(\cos b|x| + \frac{b}{a}\sin b|x|)\right]\right\}$$

The terms $\exp(-a|x|)$, $\cos b|x|$, and $\sin b|x|$ are expanded in their power series as in Section IV-D-2 to obtain

$$\exp\left[-m^2 f(t)\right] = \exp\left\{-m^2\left[1 - \exp\left(-\frac{2n\pi a}{b}\right)\left(1 - \frac{x^2}{2} + \cdots\right)\right]\right\}$$

$$= \exp\left\{-m^2\left[1 - \exp\left(-\frac{2n\pi a}{b}\right)\right]\right\}\exp\left[-\frac{m^2}{2}\exp\left(-\frac{2n\pi a}{b}\right)x^2\right]$$

when higher order terms in the resulting power series in x are neglected. Using this approximate expression for the envelope function gives

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^2}{4}\cos\frac{\omega_c t}{\omega_0}\sum_{n=0}^{\infty}\varepsilon_n \exp\left\{-m^2\left[1 - \exp\left(-\frac{2\pi na}{b}\right)\right]\right\} g(t) ,$$

where

$$g(t) = \exp\left[-\frac{m^2}{2}\exp\left(-\frac{2n\pi a}{b}\right)\left(t-\frac{2\pi n}{b}\right)^2\right] + \exp\left[-\frac{m^2}{2}\exp\left(-\frac{2\pi n a}{b}\right)\left(t+\frac{2\pi n}{b}\right)^2\right]$$

and

$$\epsilon_n = 1 \quad \text{for} \quad n = 0 \quad \text{and} \quad \epsilon_n = 2 \quad \text{for} \quad n \neq 0 \ .$$

For cases of interest $b = \sqrt{1 - \frac{1}{4Q^2}}$ is very nearly equal to unity since values of Q on the order of ten or more will be considered. The power density spectrum is found to be[6]

$$S_v^+(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{C_0^2}{\Delta\omega}\sum_{n=0}^{\infty}\epsilon_n \exp\left(\frac{n\pi}{2Q}\right)\exp\left\{-m^2\left[1-\exp\left(-\frac{n\pi}{Q}\right)\right]\right\}\times$$

$$\exp\left[-\frac{\beta^2}{2m^2}\exp\left(\frac{\pi n}{Q}\right)\right]\cos 2\pi n\beta \quad ,$$

where $\beta = \frac{\omega-\omega_c}{\omega_0}$ as before. Relatively few terms of this power series need be computed to determine the shape of the power spectrum. For example, the curves shown in Figure 6 were obtained using only the first three terms. The accuracy of the approximation is readily apparent when compared with the digitally computed curves in Figure 5.

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[6]See Appendix C.

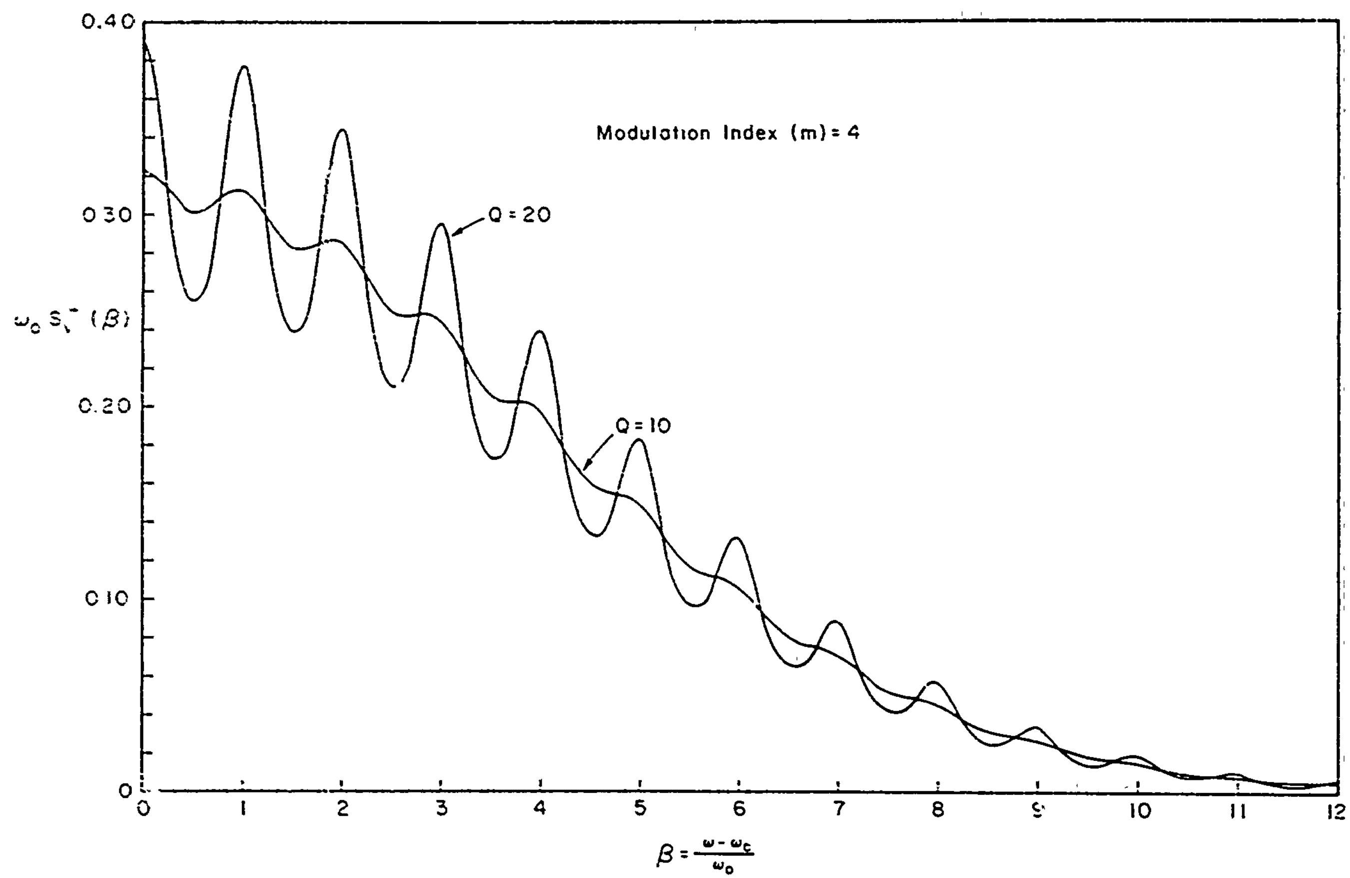


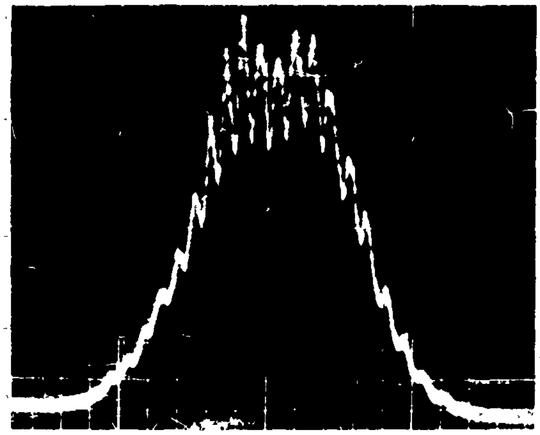
FIGURE 6 FM BY BAND-PASS NOISE POWER DENSITY SPECTRA (APPROXIMATION)

As a smooth spectrum condition is approached by increasing m with Q fixed, the major contribution to the deviation of the spectrum from the limiting Gaussian spectrum is caused by the first maximum in the envelope of the correlation function on each side of $\tau = 0$, that is, for $\tau = \pm \frac{2\pi}{\omega_0}$. This corresponds to the $n = 1$ term in the power series expression for $S_v^+(\beta)$. Also the maximum deviation for the Gaussian limiting spectrum occurs at $\beta = 0$. Thus

$$\Delta S_v^+(\beta)_{max} = 2\exp\left(\frac{\pi}{2Q}\right)\exp\left\{-m^2\left[1 - \exp\left(-\frac{\pi}{Q}\right)\right]\right\},$$

where $\Delta S^+_{v\ max}$ is the maximum deviation from the Gaussian spectrum.

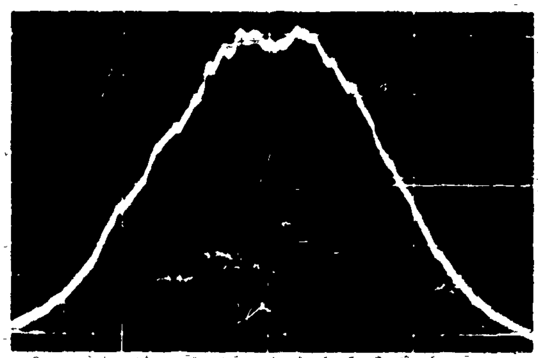
If $\Delta S_v^+(\beta)_{max}$ is fixed this expression provides a means of determining the required m for a given Q. Two experimentally determined power spectra are shown in Figure 7 illustrating the appearance of the spectrum before and after smoothing takes place. The data presented in Figure 8 were determined by observing the minimum m required to produce an arbitrarily smooth spectrum for various values of Q. The solid line was computed by use of the relation between m and Q for $\Delta S_v^+(\beta)_{max} = 1$ per cent.



(a) Power Spectrum Before Smoothing

m = 4.0 f_0 = 12.15mc f_1 = 0.6mc

Horizontal Scale 20mc/division



(b) Power Spectrum After Smoothing

m = 6.5 f_0 = 12.0mc f_1 = 0.6mc

Horizontal Scale 20mc/division

FIGURE 7 FM BY BAND-PASS NOISE POWER DENSITY SPECTRA (EXPERIMENTAL)

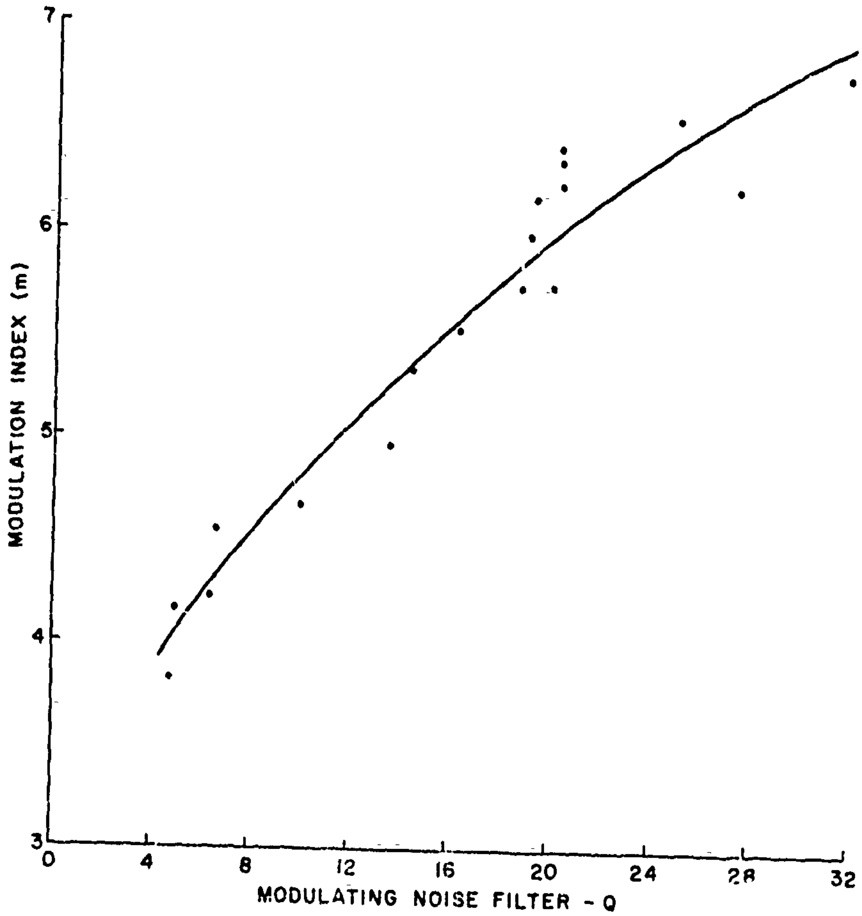


FIGURE 8 MODULATION INDEX VERSUS MODULATING NOISE Q FOR A SMOOTH SPECTRUM CONDITION

V. FM BY BAND-PASS NOISE AND A SINUSOID

In this chapter the approximation technique used to obtain the power density spectrum of FM by band-pass noise will be employed to study the case of FM by a waveform which consists of the linear sum of band-pass Gaussian noise and a sinusoid. We start with the general expression for the FM correlation function, that is

$$R_v(\tau) = \frac{C_0^2}{2} \left[\cos\left(\omega_c \tau + \phi(t_2) - \phi(t_1)\right)\right] \quad ,$$

where the modulation function $\phi(t)$ is now

$$\begin{aligned}\phi(t) &= D \int_{-\infty}^{t} \left[v_n(t') - E_m \sin(\omega_0 t' + a)\right] dt' \\ &= D \int_{-\infty}^{t} v_n(t')\, dt' + \frac{DE_m}{\omega_0} \cos(\omega_0 t + a) \quad .\end{aligned}$$

Thus, the modulation function is seen to consist of a term due to the noise identical to that employed in the FM by band-pass noise only case, and a term due to the sinusoid. Let

$$\psi(t) = \int_{-\infty}^{t} v_n(t')\, dt'$$

and

$$r(t) = m_s \cos(\omega_0 t + a) \quad ,$$

where $m_s = \frac{DE_m}{\omega_0}$ is the modulation index corresponding to the

sinusoid and α is a random phase angle that is uniformly distributed over 0 to 2π. Note that m_s is defined in the usual way for FM by a sinusoid, that is, the ratio of peak deviation and modulating frequency. The correlation function can now be written

$$R_V(\tau) = \frac{C_0^{\,2}}{2} R_e \left\{ \exp(j\omega_c\tau)\ E\left[\exp(jr_2 - jr_1)\exp(j\psi_2 - j\psi_1)\right]\right\}$$

$$= \frac{C_0^{\,2}}{2} R_e \left\{ \exp(j\omega_c\tau)\ E\left[\exp(jr_2 - jr_1)\right]\ E\left[\exp(j\psi_2 - j\psi_1)\right]\right\}$$

with the last step being justified by the fact that the expected value of the product of two independent random variables is equal to the product of the expected values. Note that the shortened notation $r(t_2) = r_2$, etc. has been employed.

This expression for the correlation function is the same as that obtained for FM by band-pass noise only, except for the additional expected value term due to the sinusoid. This expected value will now be evaluated.

$$E\left[\exp(jr_2 - jr_1)\right] = E\left[\exp(jr_2)\exp(-jr_1)\right]$$

$$= E\left\{\exp\left[jm_s\cos(\omega_0 t_2 + \alpha)\right]\exp\left[-jm_s\cos(\omega_0 t_1 + \alpha)\right]\right\}.$$

The exponentials can be expanded by use of the following identity:

$$\exp(jm_s \cos \theta) = \sum_{k=0}^{\infty} \varepsilon_k\, j^k J_k(m_s) \cos k\theta \quad ,$$

where J_k is the k^{th} Bessel function of the first kind. The product is written as a double sum to obtain

$$E\left[\exp(jr_2 - jr_1)\right] =$$

$$E\left\{ \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \varepsilon_k \varepsilon_\ell\, j^k(-j)^\ell J_k(m_s) J_\ell(m_s) \cos\left[k(\omega_0 t_2 + \alpha)\right] \cos\left[\ell(\omega_0 t_1 + \alpha)\right]\right\}.$$

If the product of the cosine terms is expanded to obtain terms consisting of the cosine of the sum of the arguments and the difference of the arguments, it can be seen that the operation of carrying out the expected value integration with respect to the random phase α will eliminate the cosine sum term, with the result that $E\left[\exp(jr_2 - jr_1)\right]$ reduces to

$$E\left\{ \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \varepsilon_k \varepsilon_k\, j^k(-j)^\ell J_k(m_s) J_\ell(m_s) \cos\left[(k-\ell)\omega_0 t + k\omega_0 \tau + (k-\ell)\alpha\right]\right\}.$$

Further it is seen that this expression yields a non-zero value only when $k = \ell$.

Thus

$$E\left[\exp(jr_2 - jr_1)\right] = \sum_{k=0}^{\infty} \varepsilon_k{}^2 j^k(-j)^k J_k{}^2(m_s) \cos k\omega_0\tau$$

$$= \sum_{k=0}^{\infty} \varepsilon_k J_k{}^2(m_s) \cos k\omega_0\tau$$

$$= \sum_{k=-\infty}^{\infty} J_k{}^2(m_s) \exp(jk\omega_0\tau) \quad .$$

Using the results obtained for FM by band-pass noise only, we find that the adding of a sinusoidal modulation causes the resulting power density spectrum to become

$$S_v^+(\beta) = \frac{C_0{}^2}{2\omega_0}\sqrt{\frac{\pi}{2}} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n K_n J_k{}^2(m_s) \exp\left[-\frac{(\beta-k)^2}{2\,a_n}\right] \cos\left[t_n(\beta-k)\right],$$

where the notation of Appendix C has been used.

A relation between the modulation index, m, which corresponds to the noise modulation, and m_s, which corresponds to the sinusoidal modulation, can be obtained in terms of the ratio of the sinusoid to noise average power. Let

$$p = \frac{\overline{E_m{}^2 \sin^2(\omega_0 t + a)}}{\overline{v_n{}^2(t)}} = \frac{E_m{}^2}{2\,\overline{v_n{}^2}} = \frac{\dfrac{D^2 E_m{}^2}{\omega_0{}^2}}{\dfrac{2D^2\,\overline{v_n{}^2}}{\omega_0{}^2}} = \frac{m_s{}^2}{2m^2}$$

or

$$m = \frac{m_s}{\sqrt{2}\,p} \quad .$$

The power density spectrum can now be expressed in terms of the parameters, m_s, p, and Q.

$$S_v^+(\beta) = \frac{C_0^2\sqrt{\pi}\,p}{2\omega_0 m_s} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \epsilon_n J_k^2(m_s) \exp\left(\frac{\pi n}{2Q}\right) \exp\left\{-\frac{m_s^2}{2p}\left[1 - \exp\left(-\frac{\pi n}{Q}\right)\right]\right\} \times$$

$$\exp\left[-\frac{p(\beta-k)^2}{m_s^2} \exp\left(\frac{\pi n}{2Q}\right)\right] \cos\left[2\pi n\,(\beta-k)\right] \quad ,$$

where, as in Chapter IV, the assumption has been made that $b = \sqrt{1 - \frac{1}{4Q^2}} \approx 1$.

This expression, although rather formidable at first glance, does yield an insight into the spectral shape that is obtained from FM by band-pass noise and a sinusoid. The spectrum is seen to be made up of the sum of a set of FM by band-pass noise spectra which are centered at integer values along the β axis with each having a peak value corresponding to $J_k^2(m_s)$. The degree to which each component approaches a smooth Gaussian curve will depend on the modulation index, m, or equivalently $\frac{m_s}{\sqrt{2}\,p}$. Suppose p is fixed; then the value of m_s required to produce smooth Gaussian shaped components of the resulting spectrum can be determined from the results of Chapter IV. In the limit, as m_s becomes large, the spectrum assumes a shape given by the probability density function of the sum of the sinusoid and the band-pass noise.

VI. SIMULTANEOUS FM BY BAND-PASS NOISE AND AM BY LOW-PASS NOISE

The waveform to be considered in this chapter consists of

$$v(t) = C(t) \cos\left[\omega_c t + \phi(t)\right] ,$$

where

$$C(t) = C_0 + v_a(t)$$

and

$$\phi(t) = D \int_{-\infty}^{t} v_n(t')\, dt' .$$

The voltage $v_n(t)$ is band-pass noise as before, while $v_a(t)$ consists of noise which is obtained by passing white noise through a low-pass RC filter.

The power density spectrum of the low-pass noise, $v_a(t)$, is

$$S_{v_a}(\omega) = \frac{2\,\omega_a\,\overline{v_a^2}}{\omega^2 + \omega_a^2} ,$$

where $\omega_a = \frac{1}{RC}$. As in the band-pass noise case the spectrum has been adjusted so that a constant rms voltage, $\sqrt{\overline{v_a^2(t)}}$, is obtained independent of ω_a. Note that ω_a is the conventional 3 db bandwidth of the noise processing RC filter. The correlation function of $v_a(t)$

is easily obtained by taking the inverse Fourier transform of $S_{v_a}(\omega)$ to get

$$R_{v_a}(\tau) = \overline{v_a^2} \exp(-\omega_a |\tau|) .$$

In Chapter III the correlation function of simultaneous FM by band-pass noise and AM by low-pass noise was found to be

$$R_v(\tau) = \frac{1}{2} E\left\{C(t_1) C(t_2) \cos\left[\omega_c\tau + \phi(t_2) - \phi(t_1)\right]\right\} .$$

Substituting for the amplitude modulating function yields

$$R_v(\tau) = \frac{1}{2} E\left\{\left[C_0^2 + C_0 v_a(t_1) + C_0 v_a(t_2) + v_a(t_2) v_a(t_1)\right] \times \right.$$

$$\left. \cos\left[\omega_c\tau + \phi(t_2) - \phi(t_1)\right]\right\} .$$

Using the fact that $v_a(t)$ has a zero mean value, and the assumption that $v_a(t)$ and $v_n(t)$ are independent gives

$$R_v(\tau) = \frac{1}{2}\left\{C_0^2 + E\left[v_a(t_2)\, v_a(t_1)\right]\right\} E\left\{\cos\left[\omega_c\tau + \phi(t_2) - \phi(t_1)\right]\right\} .$$

The second expected value is again identical with that obtained in Chapter IV, while the term $E\left[v_a(t_2)\, v_a(t_1)\right]$ is simply $R_{v_a}(\tau)$.

Using the notation of Section IV-E, we get

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^{\ 2}}{2}\left[1 + \mu \exp\left(-\frac{\omega_a|t|}{\omega_0}\right)\right]\exp\left[-m^2 f(t)\right]\cos\frac{\omega_c t}{\omega_0} \quad ,$$

where $\mu = \frac{\overline{v_a^{\ 2}}}{\overline{C_0^{\ 2}}}$ is a measure of the percentage amplitude modulation.

The power density spectrum of simultaneous FM by band-pass noise and AM by low-pass noise is thus seen to consist of two terms: one which is identical with that of FM by band-pass noise only; and one which results from a correlation function that is modified by the factor $\mu \exp\left(-\frac{\omega_a|t|}{\omega_0}\right)$.

The effect of simultaneous amplitude modulation can be readily estimated by considering the influence of the factor $\exp\left(-\frac{\omega_a|t|}{\omega_0}\right)$ on the previous derivation of the spectrum of FM by band-pass noise only. Recall that the correlation function of FM by band-pass noise was approximated by a set of displaced Gaussian functions whose amplitudes were adjusted to equal the maxima of the envelope of the exact correlation function at the points $t = \frac{2\pi n}{b}$. The major effect of the factor $\exp\left(-\frac{\omega_a|t|}{\omega_0}\right)$ is to reduce the peak amplitudes of the approximating functions. Thus the corresponding effect on the spectrum becomes one of reducing the magnitude of the degree to which the power density spectrum deviates from the limiting Gaussian shape. With these considerations, the results of Appendix C for FM by band-pass

noise can be readily extended to the case of simultaneous FM by band-pass noise and AM by low-pass noise to obtain

$$S_v^+(\beta) = \frac{C_0^{\,2}}{2\omega_0 m}\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty} \varepsilon_n \left[1 + \mu \exp\left(-\frac{\omega_a}{\omega_0}\frac{2\pi n}{b}\right)\right] \exp\left(\frac{\pi n a}{b}\right) \times$$

$$\exp\left\{-m^2\left[1 - \exp\left(-\frac{2\pi n a}{b}\right)\right]\right\} \exp\left[-\frac{\beta^2}{2m^2}\exp\left(\frac{\pi n a}{b}\right)\right] \cos\frac{2\pi n \beta}{b} \; .$$

VII. EXPERIMENTAL INVESTIGATION OF POWER DENSITY SPECTRA

The system used to generate the various random modulated waveforms discussed in this report is shown in Figure 9. Spectrum analysis was accomplished using a modified AN/APR-9 receiver, which is essentially a double IF frequency superheterodyne receiver with RF preselection employed for image rejection. The local oscillator klystron frequency is linearly swept between adjustable frequency limits by a mechanical drive unit. A display of the power density spectrum of the receiver input waveform, such as that shown in Figure 7, was obtained by use of an auxilliary oscilloscope with a long persistence cathode ray tube. A voltage proportional to the receiver local oscillator frequency was applied to the oscilloscope x-axis input; while the output of the receiver's narrow band second IF, after suitable detection and integration, was applied to the y-axis input of the oscilloscope. This method of rapid power spectrum measurement makes feasible an investigation of the effect of the various modulation parameters that would be difficult, if not impossible, if the conventional point-to-point power density spectrum measurement technique were used.

An accurate determination of the frequency sensitivity of the FM source (BWO) in radians/volt (denoted by D in this report) was obtained by applying a sinusoidal modulating voltage of sufficient amplitude to cause the center frequency spectral line to be zero.

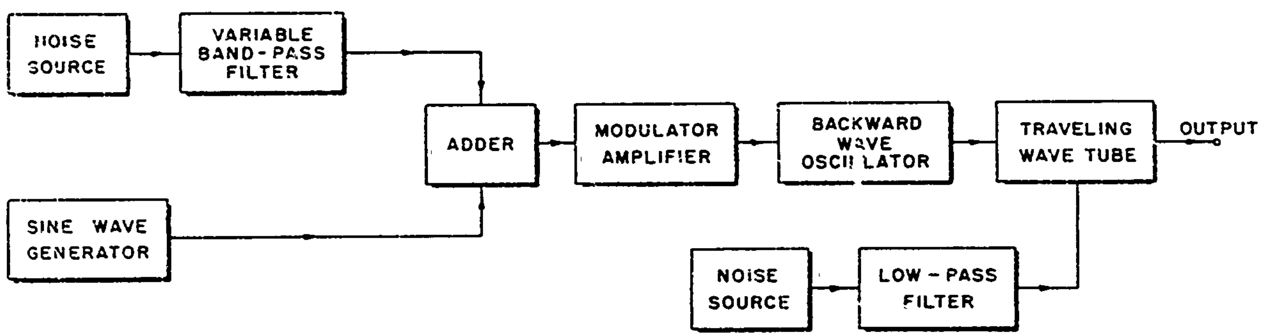


FIGURE 9 BLOCK DIAGRAM OF EXPERIMENTAL EQUIPMENT

Since the center frequency spectral line is given by $J_0^2(m_s)$, the first null of this spectral line will occur at a value of m_s corresponding to the first zero of $J_0(m_s)$. The peak magnitude, E_m, of the modulating sinusoid that is required to produce the first null of the center frequency spectral line is measured, and D is then computed from the relation $D = \frac{\omega_0 m_s}{E_m}$, where $f_0 = \frac{\omega_0}{2\pi}$ is the frequency of the modulation. A similar technique using other spectral lines having magnitudes of $J_k^2(m_s)$ can be used to determine the value of D corresponding to various modulation voltage magnitudes and thereby determine the modulation linearity of the FM source. Once D has been determined, the mean square noise voltage $\overline{v_n^2(t)}$, required to produce a desired noise modulation index, m, is computed from the relation

$$\overline{v_n^2(t)} = \frac{\omega_0^2 m^2}{D^2} ,$$

and $v_n(t)$ is adjusted to obtain the required $\overline{v_n^2}$ with the aid of a true rms meter.

APPENDIX A

If a band-pass circuit, whose transfer function is

$$H(s) = \frac{H_0 s}{s^2 + \omega_1 s + \omega_0^2} \quad ,$$

is used to filter white noise, that is, noise having a constant power density spectrum, then the resulting output voltage, $v_n(t)$, has a power density spectrum given by

$$S_{v_n}(\omega) = \frac{H_0^2 N \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega_1^2 \omega^2} \quad ,$$

where N is the constant power density level of the input noise. The mean square output voltage is

$$\overline{v_n^2} = \frac{H_0^2 N}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2 \, d\omega}{(\omega_0^2 - \omega^2)^2 + \omega_1^2 \omega^2} \quad .$$

This integral can be readily evaluated by contour integration. By analytic continuation the integral becomes

$$\int_C \frac{z^2}{(\omega_0^2 + z^2)^2 + \omega_1^2 z^2} \, dz \quad ,$$

where C is a closed contour along the real line $-\infty < \omega < \infty$ and encircling the upper half plane. The integrand has two poles that lie within the contour, one at $z = z_1 = \sqrt{\omega_0^2 - \frac{\omega_1^2}{4}} + j\frac{\omega_1}{2}$ and

another at $z = z_1^*$. The integral can be written

$$\int_C \frac{z^2\,dz}{(z-z_1)(z-z_1^*)(z+z_1)(z+z_1^*)} = 2\pi j\left[\text{Sum of Residues at } z_1 \text{ and } -z_1^*\right]$$

$$= 2\pi j\left[\frac{z_1^2}{(z_1-z_1^*)(z_1+z_1)(z_1+z_1^*)} + \frac{(-z_1^*)^2}{(-z_1^*-z_1)(-z_1^*-z_1^*)(-z_1^*+z_1)}\right]$$

$$= \pi j\,\frac{(z_1+z_1^*)}{z_1^2 - z_1^{*2}} = \frac{\pi j}{z_1 - z_1^*} = \frac{\pi j}{2j\,\mathrm{Im}\,(z_1)} = \frac{\pi}{\omega_1} \quad .$$

Note: The symbol Im denotes "imaginary part of."

The integral over the upper half z-plane circle can be easily shown to be zero in view of the fact that the degree of the denominator of the integrand is greater, by two, than the degree of the numerator. Thus the result

$$\int_{-\infty}^{\infty} \frac{\omega^2\,d\omega}{(\omega_0^2-\omega^2)^2 + \omega_1^2\,\omega^2} = \frac{\pi}{\omega_1}$$

is obtained.

APPENDIX B

In order to obtain the correlation function of FM by band-pass noise an integral of the form

$$I(\omega) = \int_{-\infty}^{\infty} \frac{(1-\cos \omega\tau)\, d\omega}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\, \omega^2}$$

must be evaluated. The integral $I(\omega)$ can be written as the sum of two integrals, $I_1(\omega)$ and $I_2(\omega)$, with

$$I_1(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\, \omega^2}$$

and

$$I_2(\omega) = \int_{-\infty}^{\infty} \frac{-\cos \omega\tau\, d\omega}{(\omega_0^2 - \omega^2)^2 + \omega_1^2\, \omega^2} .$$

These integrals are evaluated by means of contour integration. By analytic continuation, the first integral, $I_1(\omega)$, becomes

$$I_1(\omega) = \int_C \frac{dz}{(\omega_0^2 - z^2)^2 + \omega_1^2\, z^2} ,$$

where the contour is taken along the real line $-\infty < \omega < \infty$ and the upper half plane with the integral around the upper half plane being zero as was the case in Appendix A. Note that the poles of the integrand are also the same as those of the integral in Appendix A.

$$I_1(\omega) = 2\pi j \left[\text{Sum of Residues at } z_1 \text{ and } -z_1^* \right]$$

$$= \frac{\pi j}{(z_1^2 - z_1^{*2})} \left(\frac{1}{z_1} + \frac{1}{z_1^*} \right) = \pi j \frac{1}{z_1 z_1^* (z_1 - z_1^*)} = \frac{\pi}{\omega_1 |z_1|^2} = \frac{\pi}{\omega_1 \omega_0^2} .$$

The integral $I_2(\omega)$ is evaluated as follows, where the symbols Re and Im denote "real part of" and "imaginary part of" respectively:

$$-I_2(\omega) = \int_{-\infty}^{\infty} \frac{\cos \omega\tau \, d\omega}{(\omega_0^2 - \omega^2)^2 + \omega_1^2 \omega^2} = \text{Re} \int_C \frac{\exp(jz|\tau|)\, dz}{(\omega_0^2 - z^2)^2 + \omega_1^2 z^2}$$

$$= \text{Re}\left\{ \frac{\pi j}{z_1^2 - z_1^{*2}} \left[\frac{\exp(jz_1|\tau|)}{z_1} + \frac{\exp(-j z_1^* |\tau|)}{z_1^*} \right] \right\}$$

$$= \text{Re}\left\{ \frac{\pi j}{|z_1|^2 (z_1^2 - z_1^{*2})} \left[z_1^* \exp(jz_1|\tau|) + z_1 \exp(-jz_1^*|\tau|) \right] \right\}$$

$$= \text{Re}\left\{ \frac{\pi}{2|z_1|^2 \text{Re}(z_1) I_m(z_1)} \exp\left[(-I_m(z_1|\tau|) \right] \left[\text{Re}(z_1) \cos \text{Re}(z_1|\tau|) \right. \right.$$

$$\left. \left. + \text{Im}(z_1) \sin \text{Re}(z_1|\tau|) \right] \right\}$$

$$= \frac{\pi}{\omega_1 \omega_0^2} \exp\left(-\frac{\omega_1|\tau|}{2}\right) \left[\cos\left(\sqrt{\omega_0^2 - \frac{\omega_1^2}{4}} \, |\tau|\right) + \frac{\omega_1}{2\sqrt{\omega_0^2 - \frac{\omega_1^2}{4}}} \sin\left(\sqrt{\omega_0^2 - \frac{\omega_1^2}{4}} |\tau|\right) \right].$$

APPENDIX C

In Chapter IV the correlation function of FM by band-pass noise is approximated using the following expression:

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^2}{4} \cos \frac{\omega_c t}{\omega_0} \sum_{n=0}^{\infty} \epsilon_n \exp\left\{-m^2\left[1 - \exp\left(-\frac{2\pi n a}{b}\right)\right]\right\} g(t) ,$$

where

$$g(t) = \exp\left[-\frac{m^2}{2} \exp\left(-\frac{2\pi n a}{b}\right)\left(t - \frac{2\pi n}{b}\right)^2\right] + \exp\left[-\frac{m^2}{2} \exp\left(-\frac{2\pi n a}{b}\right)\left(t + \frac{2\pi n}{b}\right)^2\right].$$

Let

$$a_n^2 = m^2 \exp\left(-\frac{2\pi n a}{b}\right)$$

and

$$t_n = \frac{2\pi n}{b} ,$$

then

$$g(t) = \exp\left[-\frac{a_n^2}{2}(t - t_n)^2\right] + \exp\left[-\frac{a_n^2}{2}(t + t_n)^2\right].$$

Also let

$$K_n = \exp\left\{-m^2\left[1 - \exp\left(-\frac{2\pi n a}{b}\right)\right]\right\}.$$

Now R_v can be written

$$R_v\left(\frac{t}{\omega_0}\right) = \frac{C_0^2}{4} \cos \frac{\omega_c t}{\omega_0} \sum_{n=0}^{\infty} \epsilon_n K_n \, g(t) .$$

The power density spectrum, $S_v(\omega)$, can now be obtained by taking the Fourier transform of $R_v(\frac{t}{\omega_0})$. Thus

$$S_v(\omega) = \int_{-\infty}^{\infty} R_v(\frac{t}{\omega_0}) \exp(\frac{\omega t}{\omega_0}) \frac{dt}{\omega_0} \quad .$$

Upon writing the $\cos \frac{\omega_c t}{\omega_0}$ term contained in $R_v(\frac{t}{\omega_0})$ in exponential form, $S_v(\omega)$ becomes

$$S_v(\omega) = \frac{C_0^{\;2}}{8\omega_0} \sum_{n=0}^{\infty} \varepsilon_n K_n \left\{ \int_{-\infty}^{\infty} g(t) \exp\left[-j \frac{(\omega-\omega_c)}{\omega_0} t\right] dt + \int_{-\infty}^{\infty} g(t) \exp\left[-j \frac{(\omega-\omega_c)}{\omega_0} t\right] dt \right\} .$$

As before (Section IV-E) we assume that $\omega_c >> \omega_0$, which leads to the conclusion that the power density spectrum appears centered about the frequencies ω_c and $-\omega_c$. Since $S_v(\omega)$ is an even function, attention need only be focused on the spectral behavior around $\omega = \omega_c$ which is obtained by using only the first integral in the expression for $S_v(\omega)$. Thus $S_v^+(\beta)$, which is $S_v(\omega)$ expanded about $\omega = \omega_c$, with $\beta = \frac{\omega - \omega_c}{\omega_0}$, is

$$S_v^+(\beta) = \frac{C_0^{\;2}}{8\omega_0} \sum_{n=0}^{\infty} \varepsilon_n K_n \int_{-\infty}^{\infty} g(t) \exp(-j\beta t)\, dt \quad .$$

An asymptotic power series approximation for the power density spectrum of FM by band-pass noise is thus obtained upon evaluation of the integral

$$\int_{-\infty}^{\infty} g(t) \exp(-j\beta t)\, dt \quad .$$

Substituting for $g(t)$ leads to the two integrals

$$\int_{-\infty}^{\infty} \exp\left[-\frac{a_n^2}{2}(t+t_n)^2 - j\beta t\right] dt + \int_{-\infty}^{\infty} \exp\left[-\frac{a_n^2}{2}(t-t_n)^2 - j\beta t\right] dt \quad .$$

If the first integral is evaluated, the second integral can be readily obtained by simply changing the sign of t_n. The first integral is seen to be

$$\int_{-\infty}^{\infty} \exp\left[-\frac{a_n^2}{2}\left(t^2 + 2\left(t_n + \frac{j\beta}{a_n^2}\right) t + t_n^2\right)\right] dt \quad ,$$

which upon completion of the square becomes

$$\exp\left[-\frac{\beta^2}{2a_n^2} + j\,t_n\beta\right] \int_{-\infty}^{\infty} \exp\left[-\frac{a_n^2}{2}\left(t + t_n + \frac{j\beta}{a_n^2}\right)^2\right] dt \quad .$$

Let

$$u = t + t_n + \frac{j\beta}{a_n^2} \quad ,$$

then this integral becomes

$$\int_{-\infty}^{\infty} \exp\left[-\frac{a_n^2}{2}u^2\right] du = \frac{\sqrt{2\pi}}{a_n} \quad .$$

This last integral uses the well-known result of the integral of the Gaussian first order probability density function, where $\sigma^2 = \frac{1}{a_n^2}$ is the variance.

Thus the integral

$$\int_{-\infty}^{\infty} g(t) \exp(-j\beta t)\, dt =$$

$$\frac{\sqrt{2\pi}}{a_n} \exp\left(-\frac{\beta^2}{2a_n^2}\right) \left[\exp(j t_n \beta) + \exp(-j t_n \beta)\right] =$$

$$\frac{2\sqrt{2\pi}}{a_n} \exp\left(-\frac{\beta^2}{2a_n^2}\right) \cos t_n \beta \quad ,$$

and the power series approximation for $S_v^+(\beta)$ is

$$S_v^+(\beta) = \frac{C_0^2}{2\omega_0} \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \varepsilon_n \frac{K_n}{a_n} \exp\left(-\frac{\beta^2}{2a_n^2}\right) \cos t_n \beta$$

$$= \frac{C_0^2}{2\omega_0 m}\sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \varepsilon_n \exp\left(\frac{\pi n a}{b}\right) \exp\left\{-m^2\left[1-\exp\left(-\frac{2\pi n a}{b}\right)\right]\right\}$$

$$\exp\left[-\frac{\beta^2}{2m^2} \exp\left(\frac{\pi n a}{b}\right)\right] \cos \frac{2\pi n}{b} \beta \quad .$$

Notice that the zeroth order term of this expression is

$$\frac{C_0^2}{2\Delta\omega}\sqrt{\frac{\pi}{2}} \exp\left(-\frac{\beta^2}{2m^2}\right) \quad ,$$

which is the same as that obtained in Section IV-D-2 for the limiting case as the modulation index m becomes very large. Thus the higher ordered terms in the expression for $S_v^+(\beta)$ represent deviations of the spectrum from the Gaussian limit for intermediate values of m.

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